Université 8 Mai 45 Guelma **Faculté ST**
Deuxième année ST **Année Universitaire : 2021/2022**

Examen final de Maths 3 (22/01/2022)
Durée 01h:30 mn

Exercice 1 (1+3.5+2.5=07 points):

1) Déterminer la primitive

$$I_1 = \int \frac{dx}{\sqrt{x^2+2x+5}}$$

sachant que $\int \frac{dx}{\sqrt{x^2+a^2}} = \arg sh(\frac{x}{a})$

2) Calculer l'intégrale triple suivante $\quad I_2 = \iiint\limits_V \frac{1}{\sqrt{x^2+y^2+z^2}} dxdydz$

où $V = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leqslant z^2;\ z \geqslant 0 \text{ et } x^2+y^2+z^2 \leqslant 4\right\}$

3) Etudier la nature de l'intégrale impropre suivante

$$I_3 = \int_0^{+\infty} \frac{1}{\sqrt{t}}\, e^{-t+3} dt$$

Exercice 2 (2.5+3.5+3=09 points):

a) Résoudre les équations différentielles suivantes

1) $2y^2e^{2x}dx + 2ye^{2x}dy = 0$

2) $y'' + 9y = \frac{1}{\cos 3x}$

b) Résoudre l'EDP suivante en utilisant le changement de variables $u = 2x - y$ et $v = x - 3y$

$$\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y} = x + 2y$$

Exercice 3 (04 points):

Résoudre l'équation différentielle suivante en utilisant la transformation de Laplace

$$x''(t) - 2x'(t) + x(t) = e^{-2t}, \qquad \text{avec } x(0) = 0 \text{ et } x'(0) = 0.$$

Indication:

$\pounds(e^{-at}f(t))_{(p)} = \pounds(f(t))_{(p+a)}$; $\pounds(t^n)_{(p)} = \frac{n!}{p^{n+1}}$;

$\pounds(t^n e^{\alpha t})_{(p)} = \frac{n!}{(p-\alpha)^{n+1}}$; $\pounds(e^{\alpha t})_{(p)} = \frac{1}{p-\alpha}$

Corrigé de l'Examen final de Maths 3 (22/01/2022)

EXO n°1 ① $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{dx}{\sqrt{(x+1)^2+2^2}} = \operatorname{argsh}\left(\frac{x+1}{2}\right) + cte$

② On passe aux coordonnées sphérique : $\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$

$V' = \{(r,\theta,\varphi) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi$ et $0 \le \varphi \le \frac{\pi}{4}\}$

$|J| = r^2|\sin\varphi|$

$\displaystyle\iiint_V \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}} = \int_0^2\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{r^2\sin\varphi}{\sqrt{r^2}}\,d\varphi\,d\theta\,dr = \int_0^2 r\,dr\int_0^{2\pi} d\theta\int_0^{\frac{\pi}{4}}\sin\varphi\,d\varphi$

$\displaystyle = \frac{r^2}{2}\Big/_0^2\ \theta\Big/_0^{2\pi}\ -\cos\varphi\Big/_0^{\frac{\pi}{4}} = 2\times 2\pi\times\left(-\frac{\sqrt{2}}{2}+1\right) = 2\pi(2-\sqrt{2})$

③ $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{t}}e^{-t+3}\,dt = \int_0^1 + \int_1^{+\infty}$

$\displaystyle\int_0^1 ?$ $\quad \frac{1}{\sqrt{t}} \underset{t\to 0}{\sim} \frac{1}{t^{1/2}}$ et $e^{-t+3} \sim e^{+3}$ donc $\frac{1}{\sqrt{t}}e^{-t+3} \underset{t\to 0}{\sim} e^3\frac{1}{t^{1/2}}$

$\displaystyle\int_0^1 \frac{dt}{t^{1/2}}$ intégrale de Rieman $\alpha = \frac{1}{2} < 1 \Rightarrow CV$ $\left(k = e^3 \neq 0, \neq \infty\right)$

$\displaystyle\int_1^{+\infty} ?$ on a : $\forall t \geq 1 \Rightarrow \sqrt{t} \geq 1 \Rightarrow \frac{1}{\sqrt{t}} \leq 1$. d'où

$\displaystyle\frac{1}{\sqrt{t}}e^{-t+3} \leq e^{-t+3} \underset{t\to+\infty}{\sim} e^{-t}$

$\displaystyle\int_1^{+\infty} e^{-t}\,dt = \lim_{x\to+\infty} -e^{-t}\Big/_1^x = e^{-1} < \infty$ CV $\Rightarrow \displaystyle\int_1^{+\infty}\frac{1}{\sqrt{t}}e^{-t+3}\,dt$ CV

$\displaystyle\int_0^{+\infty} = \int_0^1 + \int_1^{+\infty} = CV + CV = CV.$

EXO n°2 ① $\underbrace{2y^2e^{2x}}_{P}dx + \underbrace{2ye^{2x}}_{Q}dy = 0$

$\displaystyle\frac{\partial P}{\partial y} = 4ye^{2x} = \frac{\partial Q}{\partial x} = 4ye^{2x}$ donc l'équation est exacte

donc il existe $\varphi(x,y)$: $\begin{cases} \dfrac{\partial\varphi}{\partial x} = 2y^2e^{2x} & \text{①} \\ \dfrac{\partial\varphi}{\partial y} = 2ye^{2x} & \text{②} \end{cases}$

de ① : $\varphi(x,y) = y^2e^{2x} + k(y)$

de ② $\displaystyle\frac{\partial\varphi}{\partial y} = 2ye^{2x} + k'(y) = 2ye^{2x} \Rightarrow k'(y) = 0 \Rightarrow k(y) = c$, $c \in \mathbb{R}$

d'où $\varphi(x,y) = y^2e^{2x} + c = cte$ (1)

② $y'' + 9y = \frac{1}{\cos 3x}$

y_h ? $\quad y'' + 9y = 0 \Rightarrow r^2 + 9 = 0 \Rightarrow r_{1,2} = \pm 3i$ (0,25) (0,25)

$y_h = c_1 \cos 3x + c_2 \sin 3x$ où $c_1, c_2 \in \mathbb{R}$. (0,5)

y_p ? $\quad y_p$ solution particulière de l'éqt $\Rightarrow y_p'' + 9y_p = \frac{1}{\cos 3x}$

$y_h = c_1 \cos 3x + c_2 \sin 3x \Rightarrow y_p = c_1(x)\cos 3x + c_2(x) \sin 3x$ (0,25)

$$\Rightarrow \begin{cases} c_1' \cos 3x + c_2' \sin 3x = 0 \\ -3c_1' \sin 3x + 3c_2' \cos 3x = \frac{1}{\cos 3x} \end{cases}$$ (0,5)

En utilisant la méthode Cramer on aura:

$$c_1' = \frac{\begin{vmatrix} 0 & \sin 3x \\ \frac{1}{\cos 3x} & 3\cos 3x \end{vmatrix}}{\begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}} = \frac{-\frac{\sin 3x}{\cos 3x}}{3\cos^2 3x + 3\sin^2 3x} = \frac{-\sin 3x}{3\cos 3x}$$ (0,5)

$3.(\cos^2 + \sin^2) = 3$

$$\Rightarrow c_1 = \frac{1}{3}\int \frac{-\sin 3x}{\cos 3x}\,dx = \frac{1}{9}\ln(\cos 3x)$$ (0,25)

de plus:

$$c_2' = \frac{\begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{\cos 3x} \end{vmatrix}}{\begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}} = \frac{1}{3} = 1 \Rightarrow c_2 = \frac{1}{3}\int dx = \frac{1x}{3}$$ (0,5) (0,25)

d'où $y_p = \frac{1}{9}\ln(\cos 3x) + \frac{1x}{3}\sin 3x$ et $y_g = y_h + y_p$ (0,25)

③ $\begin{cases} u = 2x - y \\ v = x - 3y \end{cases} \Rightarrow \frac{\partial u}{\partial x} = 2$ (0,25) $\quad \frac{\partial v}{\partial y} = -3$ (0,25)

$\frac{\partial v}{\partial x} = 1$ (0,25) $\quad \frac{\partial u}{\partial y} = -1$ (0,25) posons $f(x,y) = \tilde{f}(u,v)$

d'où: $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial \tilde{f}}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial \tilde{f}}{\partial v} = 2\frac{\partial \tilde{f}}{\partial u} + \frac{\partial \tilde{f}}{\partial v}$ (0,5)

$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}\cdot\frac{\partial \tilde{f}}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial \tilde{f}}{\partial v} = -\frac{\partial \tilde{f}}{\partial u} - 3\frac{\partial \tilde{f}}{\partial v}$ (0,5)

En remplaçant dans $\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y} = x + 2y$

$\Rightarrow -5\frac{\partial \tilde{f}}{\partial v} = u - v$ (0,25) $\Rightarrow \tilde{f}(u,v) = -\frac{1}{5}uv + \frac{1}{10}v^2 + k(u)$ (0,5)

d'où: $f(x,y) = -\frac{1}{5}(2x-y)(x-3y) + \frac{1}{10}(x-3y)^2 + k(2x-y)$ (0,25) où k fct q/q

(2)

EXO: 3

$x''(t) - 2x'(t) + x(t) = e^{-2t}$

$\Rightarrow \mathcal{L}(x''(t))(p) - 2\mathcal{L}(x'(t))(p) + \mathcal{L}(x(t))(p) = \mathcal{L}(e^{-2t})(p) \cdot (I)$ 0,25

Posons: $\mathcal{L}(x(t))(p) = F(p)$ 0,25

$\mathcal{L}(x'(t))(p) = F'(p) = pF(p) - \overset{0}{\overset{\shortparallel}{x}}(0) = pF(p)$ 0,5

$\mathcal{L}(x''(t))(p) = F''(p) = p^2F(p) - p\overset{0}{\overset{\shortparallel}{x}}(0) - \overset{0}{\overset{\shortparallel}{x'}}(0) = p^2F(p)$ 0,5

Ainsi on aura:

$(I) \Rightarrow F''(p) - 2F'(p) + F(p) = \frac{1}{p+2}$ 0,25

$\Rightarrow p^2F(p) - 2pF(p) + F(p) = \frac{1}{p+2}$

$\Rightarrow (p^2 - 2p + 1)F(p) = \frac{1}{p+2}$ 0,25

$\Rightarrow F(p) = \frac{1}{(p+2)(p-1)^2} = \frac{\alpha}{p+2} + \frac{\beta}{p-1} + \frac{\gamma}{(p-1)^2}$ 0,5

$= \frac{(\alpha+\beta)p^2 + (-2\alpha+\beta+\gamma)p + \alpha - 2\beta + 2\gamma}{(p+2)(p-1)^2}$

Par identification

$\begin{cases} \alpha + \beta = 0 \\ -2\alpha + \beta + \gamma = 0 \\ \alpha - 2\beta + 2\gamma = 1 \end{cases} \Rightarrow \begin{cases} \alpha = 1/9 \\ \beta = -1/9 \\ \gamma = 1/3 \end{cases}$ 0,75

d'où $F(p) = \frac{1}{9(p+2)} - \frac{1}{9(p-1)} + \frac{1}{3(p-1)^2}$

$\Rightarrow \mathcal{L}^{-1}(F(p)) = \frac{1}{9}\mathcal{L}^{-1}\left(\frac{1}{p+2}\right) - \frac{1}{9}\mathcal{L}^{-1}\left(\frac{1}{p-1}\right) + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{(p-1)^2}\right)$ 0,25

Ainsi on aura: $x(t) = \frac{1}{9}e^{-2t} - \frac{1}{9}e^{t} + \frac{1}{3}te^{t}$ 0,5