Chapter — Vectors and Coordinate Systems

2.1 Vectors

2.1.1 Definition:

A vector is a magnitude oriented in space. A vector with origin *A* and end *B*, denoted by: $\overrightarrow{AB}$ or $\vec{V}$, is defined by four characteristics:

- ***Its origin:*** the point of application *A* (*see Figure 2.1*).
- ***Its direction***: that of the straight line carrying the vector.
- ***Its orientation***: the indicated by the arrow.
- ***Its modulus:*** which is the length of the vector: $\|\overrightarrow{AB}\| = \|\vec{V}\|$.

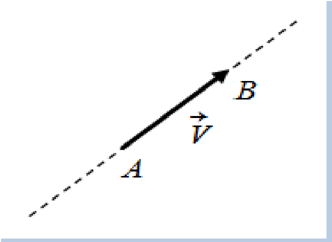


Figure 2.1: A vector

2.1.2 Unit vector

The unit vector $\vec{u}$ of a vector $\vec{V_1}$ of length $\|\vec{V_1}\| > 0$, the vector $\vec{u} = \dfrac{\vec{V_1}}{\|\vec{V_1}\|}$.

Its modulus is equal to the *unit* $\|\vec{u}\| = 1$ and its orientation is that of $\vec{V_1}$.

2.1.3 Vector Operations

Equal vectors: Two vectors $\vec{V_1}$ and $\vec{V_2}$ are equal if they have the same modulus, direction and orientation, whatever of their origin.

Opposite vectors: being $\vec{V}$ is a vector, note $-\vec{V}$ the vector opposite to $\vec{V}$. In practice $-\vec{V}$ is of opposite orientation to $\vec{V}$.

Sum or resultant: either $\vec{V_1}$ and $\vec{V_2}$ be two vectors, therefore $\vec{V_1} + \vec{V_2} = \vec{V_3}$ which is the diagonal resulting from the parallelogram formed by the vectors $\vec{V_1}$ and $\vec{V_2}$ (*see Figure 2.2*).

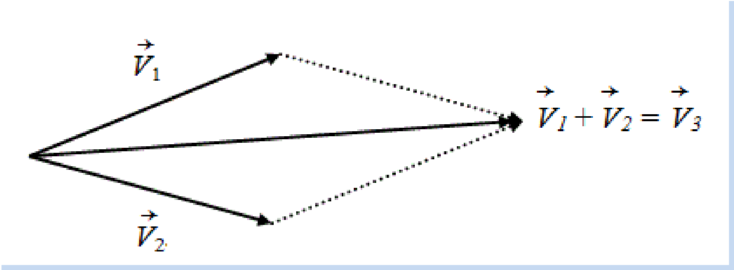


Figure 2.2: Sum of two vectors

Or: the sum of several vectors is a vector joining the origin of the first vector to the end of the last.

Subtract **:** the subtract (difference) of two vectors $\vec{V}_1$ and $\vec{V}_2$, noted $(\vec{V}_1 - \vec{V}_2)$ corresponds to adding $\vec{V}_1$ with the opposite $(-\vec{V}_2)$.

Product**:** The product of a vector $\vec{V}_1$ by a scalar p is the vector $p.\vec{V}_1$ of the same direction as $\vec{V}_1$ whose its modulus is equal to the product of that of $\vec{V}_1$ by the absolute value of p and whose orientation is identical or opposite to that of $\vec{V}_1$, depending on whether p is positive or negative (depends on the sign of p).

2.1.4 Linear Algebra Laws:

Consider the vectors $\vec{V}_1$, $\vec{V}_2$, $\vec{V}_3$ and the scalars p and q, we have:

Commutativity law	$\vec{V}_1 + \vec{V}_2 = \vec{V}_2 + \vec{V}_1$	*Commutative on addition*
Associativity Law	$\vec{V}_1 + (\vec{V}_2 + \vec{V}_3) = (\vec{V}_1 + \vec{V}_2) + \vec{V}_3$	*Associative on the addition*
Associativity Law	$p.(q.\vec{V}_1) = (p.q).\vec{V}_1 = q.(p.\vec{V}_1)$	*Associative on multiplication*
Distribution Law	$(p+q).\vec{V}_1 = p.\vec{V}_1 + q.\vec{V}_1$ $p.(\vec{V}_1 + \vec{V}_2) = p.\vec{V}_1 + p.\vec{V}_2$	*Distributive on multiplication*

2.1.6 Components of a vector in an orthonormal reference

The orthonormal reference frame in space is represented by three orthogonal axes (*Ox*, *Oy*, *Oz*) with a basis $(\vec{i}, \vec{j}, \vec{k})$.

The components of the vector $\vec{V}$ are the orthogonal projections *Vx*, *Vy* and *Vz* on the axes (*Ox*, *Oy*, *Oz*). The expression of $\vec{V}$ is given by:

$\vec{V} = Vx.\vec{i} + Vy.\vec{j} + Vz.\vec{k}$, so, its modulus is: $\|\vec{V}\| = \sqrt{Vx^2 + Vy^2 + Vz^2}$

On the other hand: $Vx = V\cos\alpha$, $Vy = V\cos\beta$, $Vz = V\cos\gamma$

Where : $\alpha = (\vec{i}, \vec{V})$, $\beta = (\vec{j}, \vec{V})$ and $\gamma = (\vec{k}, \vec{V})$

We call: $\cos\alpha$, $\cos\beta$ *and* $\cos\gamma$ ***direction cosines*** of the support of the vector $\vec{V}$ in the orthonormal reference frame (*O,xyz*).

The components of the unit vector of $\vec{V}$, noted $\vec{u}$, are :

$$\vec{u} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{Vx.\vec{i} + Vy.\vec{j} + Vz.\vec{k}}{\|\vec{V}\|} = \frac{Vx}{\|\vec{V}\|}.\vec{i} + \frac{Vy}{\|\vec{V}\|}.\vec{j} + \frac{Vz}{\|\vec{V}\|}.\vec{k}$$

$$= \cos\alpha.\vec{i} + \cos\beta.\vec{j} + \cos\gamma.\vec{k}$$

Note:

The ***direction cosines*** of the vector $\vec{V}$ are the components of its unit vector.

We deduce the relationship between the direction cosines since $\|\vec{u}\| = 1$:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

2.1.7 Scalar product

The scalar product of two vectors $\vec{V}_1$ and $\vec{V}_2$ is defined by:

$$\vec{V}_1 \cdot \vec{V}_2 = \|\vec{V}_1\| \cdot \|\vec{V}_2\| . \cos\alpha \quad ; \quad 0 \leq \alpha \leq \pi$$

Properties:

Commutative law	$\vec{V}_1 \cdot \vec{V}_2 = \vec{V}_2 \cdot \vec{V}_1$	*commutative*
Distributive law	$\vec{V}_1 \cdot (\vec{V}_2 + \vec{V}_3) = \vec{V}_1 \cdot \vec{V}_2 + \vec{V}_1 \cdot \vec{V}_3$ $p.(\vec{V}_1 \cdot \vec{V}_2) = (p.\vec{V}_1) \cdot \vec{V}_2 = \vec{V}_1 \cdot (p.\vec{V}_2)$ $= (\vec{V}_1 \cdot \vec{V}_2).p$	*Distributive*
Orthogonality	$\vec{V}_1 \cdot \vec{V}_2 = 0 \Rightarrow \vec{V}_1 \perp \vec{V}_2$	*Particular case*

In the direct basis $(\vec{i}, \vec{j}, \vec{k})$ of an orthonormal reference frame, we have:

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \qquad \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$$

Being the two vectors $\vec{V}_1$ and $\vec{V}_2$ in an orthonormal Cartesian reference:

$$\vec{V}_1 = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \quad \text{and} \quad \vec{V}_2 = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

$$\vec{V}_1 \cdot \vec{V}_1 = V_1^2 = x_1^2 + y_1^2 + z_1^2 \qquad \vec{V}_1 \cdot \vec{V}_2 = x_1x_2 + y_1y_2 + z_1z_2$$

2.1.8 Vector product

The vector product of two vectors $\vec{V_1}$ and $\vec{V_2}$ is defined by:

$$\vec{V_1} \wedge \vec{V_2} = \|\vec{V_1}\| . \|\vec{V_2}\| . \sin\alpha . \vec{u} \; ; \qquad 0 \leq \alpha \leq \pi$$

Where: $\vec{u}$ is a unit vector indicating the direction of $\vec{V_1} \wedge \vec{V_2}$.

Properties:

Non-commutative law: $\vec{V_1} \wedge \vec{V_2} = -\vec{V_2} \wedge \vec{V_1}$	*Not commutative*
Distributive law: $\vec{V_1} \wedge (\vec{V_2} + \vec{V_3}) = \vec{V_1} \wedge \vec{V_2} + \vec{V_1} \wedge \vec{V_3}$ $p(\vec{V_1} \wedge \vec{V_2}) = (p\vec{V_1}) \wedge \vec{V_2} = \vec{V_1} \wedge (p\vec{V_2})$ $= (\vec{V_1} \wedge \vec{V_2}).p$	*Distributive*
In the direct basis $(\vec{i}, \vec{j}, \vec{k})$ of an orthonormal reference frame, we have: $\vec{i} \wedge \vec{i} = \vec{j} \wedge \vec{j} = \vec{k} \wedge \vec{k} = \vec{0}\,;\ \vec{i} \wedge \vec{j} = \vec{k}\,;\ \vec{j} \wedge \vec{k} = \vec{i}\,;\ \vec{k} \wedge \vec{i} = \vec{j}$	

Being the two vectors $\vec{V_1}$ and $\vec{V_2}$ in an orthonormal Cartesian reference: $\vec{V_1} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$ and $\vec{V_2} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$ $\vec{V_1} \wedge \vec{V_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$ $= (y_1.z_2 - y_2.z_1)\vec{i} - (x_1.z_2 - x_2.z_1)\vec{j} + (x_1.y_2 - x_2.y_1)\vec{k}$	*Determinant usage*
If : $\vec{V_1} \wedge \vec{V_2} = \vec{0}$, as well as : $\vec{V_1} \neq 0$ and $\vec{V_2} \neq 0$ so: $\vec{V_1} // \vec{V_2}$	*Particular case*
$\Vert\vec{V_1} \wedge \vec{V_2}\Vert$ is the surface area of a parallelogram with sides $\vec{V_1}$ and $\vec{V_2}$	*Geometric sense*

2.1.9 Mixed product and double vector product

The mixed product of the three vectors $\vec{V_1}$,$\vec{V_2}$ and $\vec{V_3}$ is defined by:

$$\vec{V_1} .(\vec{V_2} \wedge \vec{V_3}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Where: $\vec{V_1} = x_1.\vec{i} + y_1.\vec{j} + z_1.\vec{k}$, $\vec{V_2} = x_2.\vec{i} + y_2.\vec{j} + z_2.\vec{k}$,

$\vec{V_3} = x_3.\vec{i} + y_3.\vec{j} + z_3.\vec{k}$

The absolute value of mixed product $\left|\vec{V_1} .(\vec{V_2} \wedge \vec{V_3})\right|$ represents the volume of a parallelepiped with sides $\vec{V_1}$, $\vec{V_2}$ and $\vec{V_3}$.

We also have: $\vec{V_1} .(\vec{V_2} \wedge \vec{V_3}) = \vec{V_2}.(\vec{V_3} \wedge \vec{V_1}) = \vec{V_3} .(\vec{V_1} \wedge \vec{V_2})$

The double vector product of the three vectors $\vec{V}_1$,$\vec{V}_2$ and $\vec{V}_3$ is defined by:

$$\vec{V}_1 \wedge (\vec{V}_2 \wedge \vec{V}_3) = (\vec{V}_1 . \vec{V}_3).\vec{V}_2 - (\vec{V}_1 . \vec{V}_2).\vec{V}_3$$

Like: $(\vec{V}_1 \wedge \vec{V}_2) \wedge \vec{V}_3 = (\vec{V}_1 . \vec{V}_3) . \vec{V}_2 - (\vec{V}_2 . \vec{V}_3) . \vec{V}_1$

It is clear that $\vec{V}_1 \wedge (\vec{V}_2 \wedge \vec{V}_3) \neq (\vec{V}_1 \wedge \vec{V}_2) \wedge \vec{V}_3$

2.1.10 Vector derivatives

If $\varphi(u)$ is a scalar function and if $\vec{V}(u)$ and $\vec{V}'(u)$ are vector functions, then we have:

$\dfrac{d(\varphi.\vec{V})}{du} = \varphi . \dfrac{d\vec{V}}{du} + \vec{V}.\dfrac{d\varphi}{du}$	$\dfrac{d(\vec{V}.\vec{V}')}{du} = \vec{V}.\dfrac{d\vec{V}'}{du} + \dfrac{d\vec{V}}{du} . \vec{V}'$
$\dfrac{d(\vec{V} \wedge \vec{V}')}{du} = \vec{V} \wedge \dfrac{d\vec{V}'}{du} + \dfrac{d\vec{V}}{du} \wedge \vec{V}'$	

Usual operation by keeping the scalar or vector character of the product.

2.1.11 Gradient, divergence and rotational

Consider a vector function $\vec{V} = \vec{V}(X;Y;Z)$ and a scalar function $\varphi(x, y, z)$.

Either the differential vector operator *Nabla*, defined by:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

We define the quantities:

Gradient *(Vector form)*	$\overrightarrow{grad}\,\varphi = \vec{\nabla}.\varphi = (\frac{\partial}{\partial x}.\vec{i} + \frac{\partial}{\partial y}.\vec{j} + \frac{\partial}{\partial z}.\vec{k}).\varphi$ $= \frac{\partial \varphi}{\partial x}.\vec{i} + \frac{\partial \varphi}{\partial y}.\vec{j} + \frac{\partial \varphi}{\partial z}.\vec{k}$
Divergence *(Scalar form)*	$div\vec{V} = \vec{\nabla}.\vec{V} = (\frac{\partial}{\partial x}.\vec{i} + \frac{\partial}{\partial y}.\vec{j} + \frac{\partial}{\partial z}.\vec{k}).\vec{V} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$
Rotational *(Vector form)*	$\overrightarrow{rot}\,\vec{V} = \vec{\nabla} \wedge \vec{V}$ $= (\frac{\partial}{\partial x}.\vec{i} + \frac{\partial}{\partial y}.\vec{j} + \frac{\partial}{\partial z}.\vec{k}) \wedge (X.\vec{i} + Y.\vec{j} + Z.\vec{k})$ $= (\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}).\vec{i} - (\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}).\vec{j} + (\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}).\vec{k}$

There are two remarkable relations:

$div(\overrightarrow{rot}\,\vec{V}) = \vec{\nabla}.(\vec{\nabla} \wedge \vec{V}) = 0$	$\overrightarrow{rot}\,(\overrightarrow{grad}\,\varphi) = \vec{\nabla} \wedge (\vec{\nabla}\varphi) = \vec{0}$

2.2 Coordinate systems

2.2.1 Cartesian coordinates

a) Definition

Being three orthogonal axes; $\overrightarrow{Ox}$, $\overrightarrow{Oy}$ and $\overrightarrow{Oz}$ whose vectors are $\vec{i}$, $\vec{j}$ and $\vec{k}$ respectively.

The Cartesian coordinates of a point M of space correspond to the projections of the vector $\overrightarrow{OM}$ on the three axes.

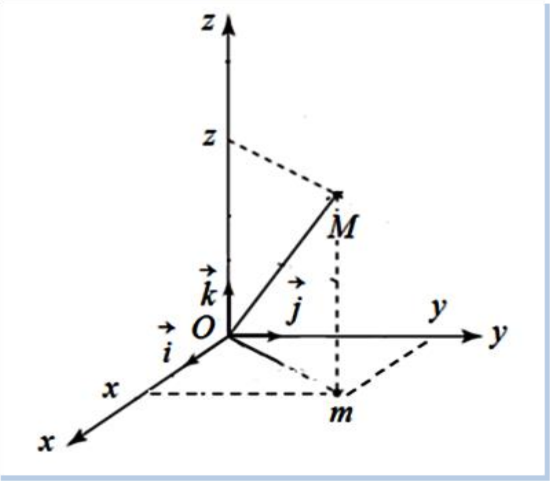


Figure 2.6: Cartesian benchmark.

The Cartesian reference frame is called normal, when the vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the same modulus.

Whereas, the benchmark is called orthonormal, if moreover, the axes *Ox*, *Oy* and *Oz* are perpendicular.

Oxyz is a direct orthonormal trihedron, if the smallest rotation that brings *Ox* onto *Oy* is in the direct trigonometric direction around *Oz*.

If $\vec{i}$, $\vec{j}$ and $\vec{k}$ form a Cartesian basis: $\overrightarrow{OM} = \vec{r} = x.\vec{i} + y.\vec{j} + z.\vec{k}$

2.2.2 Polar coordinates

a) Definition

The position of the moving point M in the plane (Oxy) is defined by the variables $\rho(t)$ and $\theta(t)$, the variables ρ, θ being the Polar coordinates of M.

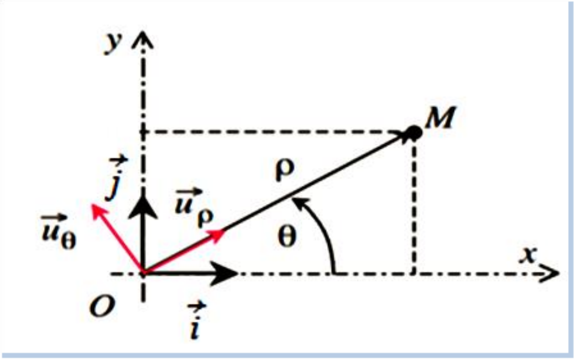


Figure 2.7: Polar benchmark.

$$\begin{cases} \rho = OM & \rho \geq 0 \\ \theta = (\overrightarrow{Ox}, \overrightarrow{OM}) & 0 \leq \theta \leq 2\pi \end{cases}$$

The reference frame defined by the basis $(\vec{u}_\rho, \vec{u}_\theta)$ is called the Polar benchmark.

$\vec{u}_\rho$ is carried by the half-line directed along the ρ crescents.

$\vec{u}_\theta$ is tangent to the trajectory following the θ crescents.

b) Relationship between Cartesian and Polar coordinates

$$\begin{cases} x = \rho.\cos\theta \\ y = \rho.\sin\theta \end{cases} \quad so: \quad \begin{cases} \rho = \sqrt{(x^2 + y^2)} \\ \theta = arc\ tan(\frac{y}{x}) \end{cases}$$

2.2.3 Cylindrical coordinates

a) Definition

The position of the *M* point is determined using the variables ρ, θ and z. These variables are called the Cylindrical coordinates of *M*.

$$M \begin{cases} \rho = Om & \rho \geq 0 \\ \theta = (\overrightarrow{Ox}, \overrightarrow{Om}) & 0 \leq \theta \leq 2\pi \\ z = Mm & -\infty \leq z \leq +\infty \end{cases}$$

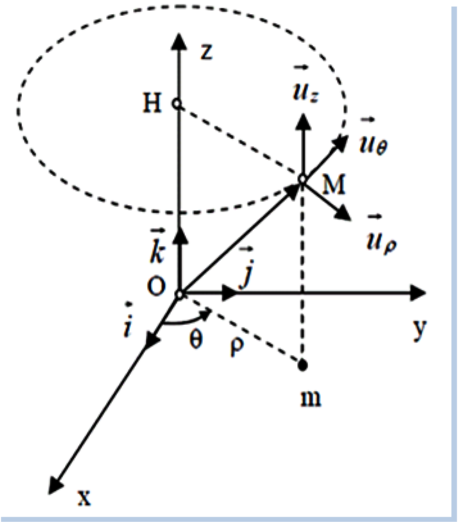


Figure 2.8: Cylindrical benchmark.

The Cylindrical basis in *M* is: $\vec{u}_\rho$, $\vec{u}_\theta$, $\vec{u}_z$.

$\vec{u}_\rho$ is carried by the half-line directed along the ρ crescents.

$\vec{u}_\theta$ is tangent to the trajectory following the θ crescents.

$\vec{u}_z$ is along the *mM*, line, in the direction of the increasing z .

b) Relationship between Cartesian and Cylindrical coordinates

$$\begin{cases} x = \rho.\cos\theta \\ y = \rho.\sin\theta \\ z = z \end{cases} \quad \text{then:} \quad \begin{cases} \rho = \sqrt{(x^2 + y^2)} \\ \theta = \arctan(\frac{y}{x}) \\ z = z \end{cases}$$

Note: the Polar reference frame is a particular case of the Cylindrical reference frame with $z = 0$.

2.3.4 Spherical coordinates

a) Definition

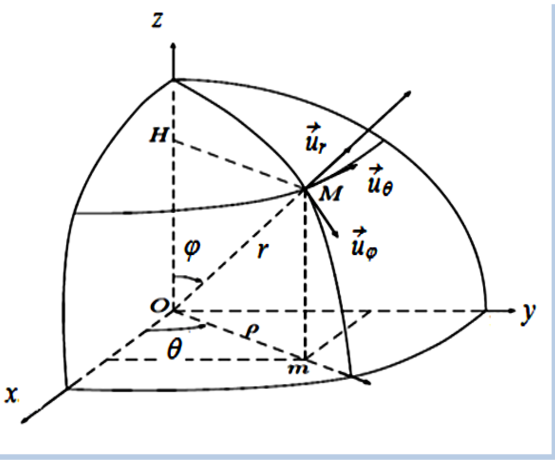


Figure 2.9: Spherical benchmark.

The position of the *M* point is determined using the variables *r*, φ and θ.

r, φ and θ are called the Spherical coordinates of *M*. φ is called *colatitude*, θ is called *azimut* or *longitude*.

$$M \begin{cases} r = OM & r \geq 0 \\ \theta = (\overrightarrow{Ox}, \overrightarrow{Om}) & 0 \leq \theta \leq 2\pi \\ \varphi = (\overrightarrow{Oz}, \overrightarrow{OM}) & 0 \leq \varphi \leq \pi \end{cases}$$

The Spherical basis in M is: $\vec{u}_r$, $\vec{u}_\varphi$, $\vec{u}_\theta$.

$\vec{u}_r$ is radial according to the ρ crescents.

$\vec{u}_\varphi$ is tangent to the vertical circle following the φ crescents.

$\vec{u}_\theta$ is tangent to the horizontal circle following the θ crescents.

b) Relationship between Cartesian and Spherical coordinates

$$\begin{cases} x = r.\sin\varphi.\cos\theta \\ y = r.\sin\varphi.\sin\theta \\ z = r.\cos\varphi \end{cases} \quad \text{then :} \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan(\frac{y}{x}) \\ \varphi = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{cases}$$

Summaries of exercises

Exercise 01

I) Being three points: *A(1,0,2)*, *B(–2,1,4)* and *C(0,3,5)* identified in an orthonormal Cartesian reference *R(O,xyz)* with a direct basis $(\vec{i}\,;\,\vec{j}\,;\,\vec{k})$.

1) Calculate the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$.
2) Deduce the unit vectors $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ respectively.
3) Calculate the vector product $\overrightarrow{AB} \wedge \overrightarrow{AC}$.
4) Deduce the surface area of the triangle *ABC*.

II) Consider a *PNQ* triangle (*see Figure 2.10 below*).

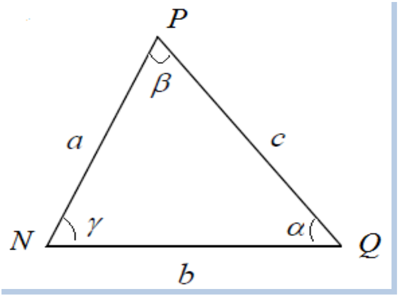


Figure 2.10

a) Demonstrate the following sines law:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

b) Demonstrate the following relationship:

$$c = \sqrt{a^2 + b^2 - 2.a.b.\cos\gamma}$$

Exercise 02

An M point of the space can be represented by:

- The Cartesian coordinates (x, y, z) of the basis $(\vec{i}, \vec{j}, \vec{k})$
- The Cylindrical coordinates (ρ, θ, z) of the basis $(\vec{u}_\rho, \vec{u}_\theta, \vec{u}_z)$
- The Spherical coordinates (r, φ, θ) of the basis $(\vec{u}_r, \vec{u}_\varphi, \vec{u}_\theta)$

1/ Express the bases $(\vec{u}_\rho, \vec{u}_\theta, \vec{u}_z)$ and $(\vec{u}_r, \vec{u}_\varphi, \vec{u}_\theta)$ as the basis $(\vec{i}, \vec{j}, \vec{k})$.

2/ Calculate:

a) $\dfrac{d\vec{u}_\rho}{d\theta}$; $\dfrac{d\vec{u}_\theta}{d\theta}$ in the basis $(\vec{i}, \vec{j}, \vec{k})$. Deduce their expressions in the Cylindrical basis $(\vec{u}_\rho, \vec{u}_\theta, \vec{u}_z)$.

b) $\dfrac{d\vec{u}_r}{d\varphi}$; $\dfrac{d\vec{u}_r}{d\theta}$; $\dfrac{d\vec{u}_\varphi}{d\varphi}$; $\dfrac{d\vec{u}_\varphi}{d\theta}$ in the Cartesian basis $(\vec{i}, \vec{j}, \vec{k})$, in the Spherical basis $(\vec{u}_r, \vec{u}_\varphi, \vec{u}_\theta)$

3/ Determine the expression of the elementary displacement vector $d\vec{L}$ in the three coordinate systems.

4/ Deduce the surface area of a disk of radius R; thus the volume of a sphere of radius r.

Exercise 03

An M point can be represented by its Cartesian coordinates (x, y, z), its Polar coordinates (ρ, θ), its Cylindrical coordinates (ρ, θ, z) or its Spherical coordinates. (r, φ, θ).

1/ The Cartesian coordinates of a point M_1 are $(2, -2)$.

Determine its Polar coordinates.

2/ The Cartesian coordinates of a point M_2 are $(2, 2, 1)$.

Determine its Cylindrical and Spherical coordinates.

3/ The Cylindrical coordinates of a point M_3 are $(3, \frac{\pi}{3} rad, 2)$.

Determine its Cartesian and Spherical coordinates.

4/ The Spherical coordinates of a point M_4 are $(3, \frac{\pi}{6} rad, \frac{\pi}{2} rad)$.

Determine its Cartesian and Cylindrical coordinates.

5/ Find the expressions and moduli of the vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_4M_3}$, in the orthonormal Cartesian basis $(\vec{i}, \vec{j}, \vec{k})$.

6/ Deduce the unit vectors of the vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_4M_3}$, in the orthonormal Cartesian basis $(\vec{i}, \vec{j}, \vec{k})$.

7/ Calculate the scalar product, as well as the vector product of two vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_4M_3}$.

Exercise 01

1/ Calculation of the modulus of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$:

We have : $\overrightarrow{AB}\begin{pmatrix}-3\\1\\2\end{pmatrix}$; so : $\|\overrightarrow{AB}\| = \sqrt{9+1+4} = \sqrt{14}$

$\overrightarrow{AC}\begin{pmatrix}-1\\3\\3\end{pmatrix}$; so : $\|\overrightarrow{AC}\| = \sqrt{1+9+9} = \sqrt{19}$

$\overrightarrow{BC}\begin{pmatrix}2\\2\\1\end{pmatrix}$; so : $\|\overrightarrow{BC}\| = \sqrt{4+4+1} = 3$

2/ Unit vectors $\vec{u}_1$; $\vec{u}_2$ and $\vec{u}_3$ of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ respectively :

$$\vec{u}_1 = \frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \frac{\sqrt{14}\,(-3\vec{i} + \vec{j} + 2\vec{k})}{14}$$

$$\vec{u}_2 = \frac{\overrightarrow{AC}}{\|\overrightarrow{AC}\|} = \frac{\sqrt{19}(-\vec{i} + 3\vec{j} + 3\vec{k})}{19}$$

$$\vec{u}_3 = \frac{\overrightarrow{BC}}{\|\overrightarrow{BC}\|} = \frac{(2\vec{i} + 2\vec{j} + \vec{k})}{3}$$

3/ Calculation of vector product $\overrightarrow{AB} \wedge \overrightarrow{AC}$:

$$\overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -3 & 1 & 2\\ -1 & 3 & 3\end{vmatrix} = -3\,\vec{i} + 7\,\vec{j} - 8\,\vec{k}.$$

4/ Deducing the surface of the triangle ABC

$$S\ (Triangle) = \frac{\|\overrightarrow{AB} \wedge \overrightarrow{AC}\|}{2} = \frac{\sqrt{122}}{2}\ \textit{units of surface.}$$

II/ a) Showing the following law: $\frac{sin\alpha}{a} = \frac{sin\beta}{b} = \frac{sin\gamma}{c}$:

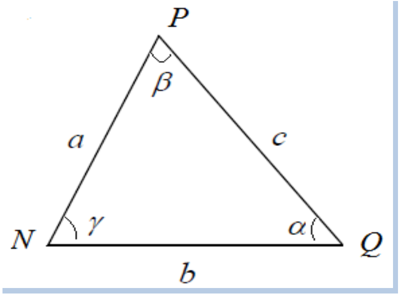


Figure 2.12

We have: $S(Triangle) = \frac{\|\overrightarrow{QP} \wedge \overrightarrow{QN}\|}{2} = \frac{c.b.sin\alpha}{2}$ (1)

$$S(Triangle) = \frac{\|\overrightarrow{PN} \wedge \overrightarrow{PQ}\|}{2} = \frac{a.c.sin\beta}{2} \quad (2)$$

$$S(Triangle) = \frac{\|\overrightarrow{NP} \wedge \overrightarrow{NQ}\|}{2} = \frac{a.b.sin\gamma}{2} \quad (3)$$

According to the three relations (1), (2) and (3), we get the sines law :

$$\frac{sin\alpha}{a} = \frac{sin\beta}{b} = \frac{sin\gamma}{c} .$$

b) Demonstration of the following relation:

$$\boldsymbol{c = \sqrt{a^2 + b^2 - 2.a.b.cos\gamma}}$$

Based on *Figure 2.12* above and the *Chasles relation*, we have:

$$\overrightarrow{PN} + \overrightarrow{NQ} = \overrightarrow{PQ} \Rightarrow \overrightarrow{PQ} = \overrightarrow{NQ} - \overrightarrow{NP}$$

So: $(\overrightarrow{PQ})^2 = (\overrightarrow{NQ} - \overrightarrow{NP})^2 \Rightarrow PQ^2 = NQ^2 + NP^2 - 2.(\overrightarrow{NQ}.\overrightarrow{NP})$

$$PQ^2 = NQ^2 + NP^2 - 2.(NQ.NP.cos\gamma)$$

We obtain: $c^2 = a^2 + b^2 - 2.a.b.cos\gamma$

Therefore: $c = \sqrt{a^2 + b^2 - 2.a.b.cos\gamma}$

Exercise 02

1/ Expression of the Cylindrical basis ($\vec{u}_\rho, \vec{u}_\theta, \vec{u}_z$) in the function of the Cartesian basis ($\vec{i}, \vec{j}, \vec{k}$) :

$$\begin{cases} \vec{u}_\rho = \cos\theta.\ \vec{i} + \sin\theta.\ \vec{j} \\ \vec{u}_\theta = -\sin\theta.\ \vec{i} + \cos\theta.\ \vec{j} \\ \vec{u}_z = \vec{k} \end{cases}$$

The Spherical basis expression ($\vec{u}_r, \vec{u}_\varphi, \vec{u}_\theta$) in the function of the Cartesian basis ($\vec{i}, \vec{j}, \vec{k}$) :

$$\begin{cases} \vec{u}_r = \cos\theta.\sin\varphi\ \vec{i} + \sin\theta.\sin\varphi.\ \vec{j} + \cos\varphi.\vec{k} \\ \vec{u}_\theta = -\sin\theta.\ \vec{i} + \cos\theta.\ \vec{j} \\ \vec{u}_\varphi = \cos\theta.\cos\varphi\ \vec{i} + \sin\theta.\cos\varphi.\ \vec{j} - \sin\varphi.\vec{k} \end{cases}$$

2/ Calculation of:

a) $\frac{d\vec{u}_\rho}{d\theta}$; $\frac{d\vec{u}_\theta}{d\theta}$ in the Cartesian basis ($\vec{i}, \vec{j}, \vec{k}$) :

$$\frac{d\vec{u}_\rho}{d\theta} = -\sin\theta.\ \vec{i} + \cos\theta.\ \vec{j}$$

$$\frac{d\vec{u}_\theta}{d\theta} = -(\cos\theta.\ \vec{i} + \sin\theta.\ \vec{j})$$

Their expressions are deduced from the Cylindrical basis ($\vec{u}_\rho, \vec{u}_\theta, \vec{u}_z$) :

$$\frac{d\vec{u}_\rho}{d\theta} = -\vec{u}_\theta$$

$$\frac{d\vec{u}_\theta}{d\theta} = -\vec{u}_\rho$$

b) $\frac{d\vec{u}_r}{d\varphi}$; $\frac{d\vec{u}_r}{d\theta}$; $\frac{d\vec{u}_\varphi}{d\varphi}$; $\frac{d\vec{u}_\varphi}{d\theta}$ **in the Cartesian basis ($\vec{i}, \vec{j}, \vec{k}$) as well as in the Spherical basis ($\vec{u}_\rho, \vec{u}_\varphi, \vec{u}_\theta$) :**

$$\frac{d\vec{u}_r}{d\varphi} = cos\theta.cos\varphi\ \vec{i} + sin\theta.cos\varphi.\ \vec{j} - sin\varphi.\vec{k} = \vec{u}_\varphi$$

$$\frac{d\vec{u}_r}{d\theta} = -\ sin\theta.sin\varphi\ \vec{i} + cos\theta.sin\varphi.\ \vec{j} = sin\varphi\ (-\ sin\theta.\ \vec{i} + cos\theta.\ \vec{j}) = sin\varphi.\ \vec{u}_\theta$$

$$\frac{d\vec{u}_\varphi}{d\varphi} = -\ cos\theta.sin\varphi\ \vec{i} - sin\theta.sin\varphi.\ \vec{j} - cos\varphi.\vec{k} = -\ \vec{u}_r$$

$$\frac{d\vec{u}_\varphi}{d\theta} = -\ sin\theta.cos\varphi\ \vec{i} + cos\theta.cos\varphi.\ \vec{j} = cos\varphi.(-\ sin\theta.\ \vec{i} + cos\theta.\ \vec{j}) = cos\varphi.\ \vec{u}_\theta$$

3/ Determining the expression of the elementary displacement vector $d\vec{L}$

In the Cartesian coordinates system:

The position vector is defined by: $\overrightarrow{OM} = x.\ \vec{i} + y.\ \vec{j} + z.\ \vec{k}$

The point M moves in space, if we derive the expression of the position vector, we will have the following elementary displacement vector expression:

$$d\overrightarrow{OM} = d\vec{L} = dx.\ \vec{i} + dy.\ \vec{j} + dz.\ \vec{k}$$

In the Cylindrical coordinates system:

By definition, the position vector is: $\overrightarrow{OM} = \rho.\ \vec{u}_\rho + z.\ \vec{u}_z$

Similarly, if we derive the expression of the position vector, we will have the elementary displacement vector expression, so:

$$d\overrightarrow{OM} = d\vec{L} = d\rho.\ \vec{u}_\rho + \rho.d\theta.\ \vec{u}_\theta + dz.\ \vec{u}_z$$

In the Spherical coordinates system:

The position vector is defined by: $\overrightarrow{OM} = r.\,\vec{u}_r$.

Then, the expression of the elementary displacement vector, will be:

$$\overrightarrow{dOM} = \overrightarrow{dL} = dr.\,\vec{u}_r + r.\sin\varphi.\,d\theta.\,\vec{u}_\theta + r.d\varphi.\,\vec{u}_\varphi$$

4/ Deduction of the surface of disk of radius *R*:

The expression of the surface element in Polar coordinates is as follows:

$dS = \rho.d\rho.d\theta$

Then: $S_{(disk)} = \int\int dS = \int\int \rho.d\rho.d\theta$

So: $S_{(disk)} = \int_{\rho=0}^{\rho=R} d\rho.\int_{\theta=0}^{\theta=2.\pi} d\theta = \frac{R^2}{2}.2\pi = \pi.R^2$

Deduction of the volume of a sphere of radius *r* :

The expression of the volume element in Spherical coordinates is as follows:

$dV = r^2.dr.\sin\varphi.d\varphi.d\theta$

We integrate dV, then : $V_{(sphere)} = \int\int\int dV = \int\int\int r^2.dr.\sin\varphi.d\varphi.d\theta$

$$V_{(sphere)} = \int_{r=0}^{r=r} r^2.dr.\int_{\varphi=0}^{\varphi=\pi} \sin\varphi.d\varphi.\int_{\theta=0}^{\theta=2.\pi} d\theta$$

On obtain : $V_{(sphere)} = \left[\frac{r^3}{3}\right]_0^r.\,[-\cos\varphi]_0^\pi.\,[\theta]_0^{2.\pi}$

$$V_{(sphere)} = \frac{4}{3}.\pi.r^3$$

Exercise 03

1/ The Polar coordinates (ρ, θ) of point $M_1(x = 2, y = -2)$:

We know that:

$\rho = \sqrt{x^2 + y^2} = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-2}{2}\right) = -45° = -\frac{\pi}{4} rad$

So, the Polar coordinates of the point M_1 are $M_1(\rho = 2\sqrt{2},\ \theta = -\frac{\pi}{4} rad)$.

2/ The Cylindrical (ρ, θ, z) and Spherical coordinates (r, φ, θ) of the point $M_2(x = 2, y = 2, z = 1)$:

– **Cylindrical coordinates (ρ, θ, z):**

We know that:

$$\rho = \sqrt{x^2 + y^2} = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{2}{2}\right) = 45° = \frac{\pi}{4} rad$$

$$z = z = 1$$

So, the Cylindrical coordinates of the point M_2 are

$$M_2(\rho = 2\sqrt{2},\ \theta = 45°, z = 1).$$

– **Spherical coordinates (r, φ, θ):**

We know that:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$\varphi = cos^{-1}\left(\frac{z}{r}\right) = cos^{-1}\left(\frac{1}{3}\right) = 70{,}53°$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{2}{2}\right) = 45° = \frac{\pi}{4} rad$$

So, the Spherical coordinates of the point M_2 are

$$M_2(r = 3,\ \varphi = 70{,}53°,\ \theta = 45°).$$

3/ The Cartesian (x, y, z) and Spherical coordinates (r, φ, θ) of the point $M_3(\rho = 3,\ \theta = \frac{\pi}{3} rad, z = 2)$:

– **Cartesian Cylindrical coordinates (x, y, z):**

We know that:

$$x = \rho.cos\theta = 3.cos\frac{\pi}{3} = \frac{3}{2} = 1{,}5$$

$$y = \rho.\sin\theta = 3.\sin\frac{\pi}{3} = 2,6$$

$$z = z = 2$$

So, the Cartesian coordinates of the point M_3 are

$$M_3(x = 1,5, y = 2,6, z = 2).$$

– **Spherical coordinates (r, φ, θ):**

We know that:

$$r = \sqrt{\rho^2 + z^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\varphi = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{2}{\sqrt{13}}\right) = 56,31°$$

$$\theta = \theta = \frac{\pi}{3} rad = 60°$$

So, the Spherical coordinates of the point M_3 are

$$M_3(r = \sqrt{13}, \varphi = 56,31°, \theta = 60°).$$

4/ The Cartesian coordinates (x, y, z) and Cylindrical (ρ, θ, z) of the point $M_4(r = 3, \varphi = \frac{\pi}{6} rad, \theta = \frac{\pi}{2} rad)$:

– **Cartesian coordinates (x, y, z):**

We know that :

$$x = r.\sin\varphi.\cos\theta = 3.\sin\frac{\pi}{6}.\cos\frac{\pi}{2} = 0$$

$$y = r.\sin\varphi.\sin\theta = 3.\sin\frac{\pi}{6}.\sin\frac{\pi}{2} = \frac{3}{2} = 1,5$$

$$z = r.\cos\varphi = 3.\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2} = 2,6$$

Then, the Cartesian coordinates of the point M_4 are

$$M_4(x = 0, y = \frac{3}{2}, z = \frac{3\sqrt{3}}{2}).$$

– **The Cylindrical coordinates (ρ, θ, z):**

We know that :

$$\rho = \sqrt{x^2 + y^2} = \sqrt{0^2 + \left(\frac{3}{2}\right)^2} = \frac{3}{2}$$

$$\theta = \theta = \frac{\pi}{2} rad = 90°$$

$$z = z = \frac{3\sqrt{3}}{2}$$

So, the Cylindrical coordinates of the point M_4 are

$$M_4(\rho = \frac{3}{2}, \theta = 90°, z = \frac{3\sqrt{3}}{2}).$$

5/ The vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_4M_3}$ in the Cartesian basis:

We have : $M_1(2, -2, 0)$, $M_2(2, 2, 1)$, $M_3(1{,}5, 2{,}6, 2)$, $M_4(0, 1{,}5, 2{,}6)$.

Therefore : $\overrightarrow{M_1M_2}\begin{pmatrix} 2-2 \\ 2-(-2) \\ 1-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$

And $\overrightarrow{M_4M_3}\begin{pmatrix} 1{,}5-0 \\ 2{,}6-1{,}5 \\ 2-2{,}6 \end{pmatrix} = \begin{pmatrix} 1{,}5 \\ 1{,}1 \\ -0{,}6 \end{pmatrix}$

The modulus of the vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_4M_3}$:

$$\|\overrightarrow{M_1M_2}\| = \sqrt{0^2 + 4^2 + 1^2} = \sqrt{17}$$

$$\|\overrightarrow{M_4M_3}\| = \sqrt{1{,}5^2 + 1{,}1^2 + (-0{,}6)^2} = 1{,}95$$

6/ The unit vectors of the vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_4M_3}$ in the Cartesian basis:

By definition, we have:

$$\vec{u}(\overrightarrow{M_1M_2}) = \frac{\overrightarrow{M_1M_2}}{\|\overrightarrow{M_1M_2}\|} = 0{,}97\,\vec{j} + 0{,}24\,\vec{k}$$

$$\vec{u}(\overrightarrow{M_4M_3}) = \frac{\overrightarrow{M_4M_3}}{\|\overrightarrow{M_4M_3}\|} = 0{,}77\,\vec{i} + 0{,}56\,\vec{j} - 0{,}31\,\vec{k}$$

7/ Scalar product and vector product of vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_4M_3}$:

– **The Scalar product: $\overrightarrow{M_1M_2} \cdot \overrightarrow{M_4M_3}$**

By definition, we have:

$$\overrightarrow{M_1M_2} \cdot \overrightarrow{M_4M_3} = x_1x_2 + y_1y_2 + z_1z_2$$

Therefore: $\overrightarrow{M_1M_2} \cdot \overrightarrow{M_4M_3} = (4 \times 1{,}1) + (1 \times (-0{,}6)) = 3{,}8$

– **The vector product: $\overrightarrow{M_1M_2} \wedge \overrightarrow{M_4M_3}$**

By definition, we have:

$$\overrightarrow{M_1M_2} \wedge \overrightarrow{M_4M_3} = (y_1.z_2 - y_2.z_1)\,\vec{i} - (x_1.z_2 - x_2.z_1)\,\vec{j} + (x_1.y_2 - x_2.y_1)\,\vec{k}$$

Therefore:

$$\overrightarrow{M_1M_2} \wedge \overrightarrow{M_4M_3} = (4 \times (-0{,}6) - 1{,}1 \times 1)\,\vec{i} - (-1{,}5 \times 1)\,\vec{j} + (-1{,}5 \times 4)\,\vec{k}$$

$$\overrightarrow{M_1M_2} \cdot \overrightarrow{M_4M_3} = -1{,}3\,\vec{i} + 1{,}5\,\vec{j} - 6\,\vec{k}$$

Chapter	*Kinematics of the Material Point*

3. Kinematics of the Material Point

3.1 General Definitions

3.1.1 Kinematics

Kinematics is the study of motion as function of time, independently of the provoking causes (*the forces applied to the material point*).

Kinematics must give the law of motion that allows defining the position of the mobile and its state of movement at each moment, so it aims to specify trajectories and time laws.

3.1.2 Benchmark

It is necessary to define a space benchmark, to track the position of a particle. This consists of choosing an origin O and a basis $(\vec{i}, \vec{j}, \vec{k})$. The trihedron $(\vec{i}, \vec{j}, \vec{k})$ is the benchmark of space.

3.1.3 Reference

A reference frame is a spatial benchmark with a time benchmark (benchmark + clock). It is therefore an object in relation to which motion is studied.

There are several references frames such as: the *Galilean* reference, the *Terrestrial* (*Earth)* reference, the *Geocentric* reference, *Kepler's or Heliocentric* reference, *Copernicus'* reference, and the *Barycentric* reference.

3.1.4 Material Point

A material point is a physical body of mass *m* whose dimensions are negligible compared to the distance over which its motion is considered.

3.1.5 Trajectory

The trajectory of a moving material point, in a given reference frame, is the curve formed by the set of successive positions occupied by the material point over time.

3.2 Kinematics without change of reference

3.2.1 Expressions of position, velocity and acceleration vectors in different coordinate systems

3.2.1.1 In the Cartesian coordinate system of the basis $(\vec{i}, \vec{j}, \vec{k})$

- *Position Vector*: $\overrightarrow{OM} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$.

Its modulus: $\|\overrightarrow{OM}\| = \sqrt{x^2 + y^2 + z^2}$

- *Velocity vector*: $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$

$$= v_x\,\vec{i} + v_y\,\vec{j} + v_z\,\vec{k}$$

Its modulus: $\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

- *Acceleration vector:*: $\vec{\gamma} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$

$$= \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j} + \frac{dv_z}{dt}\vec{k} = \gamma_x\vec{i} + \gamma_y\vec{j} + \gamma_z\vec{k}$$

Its modulus: $\|\vec{\gamma}\| = \sqrt{\gamma_x^2 + \gamma_y^2 + \gamma_z^2}$

3.2.1.2 In the Polar coordinates system of the basis $(\vec{u}_\rho, \vec{u}_\theta)$

- *Position Vector:* $\overrightarrow{OM} = \rho.\vec{u}_\rho$

Its modulus: $\|\overrightarrow{OM}\| = \rho$

- *Velocity vector:* $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \dot{\rho}.\vec{u}_\rho + \rho.\dot{\theta}.\vec{u}_\theta = v_\rho.\vec{u}_\rho + v_\theta.\vec{u}_\theta$

Its modulus: $\|\vec{v}\| = \sqrt{v_\rho^2 + v_\theta^2} = \sqrt{\dot{\rho}^2 + (\rho.\dot{\theta})^2}$

- *Acceleration vector:* $\vec{\gamma} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = (\ddot{\rho} - \rho\dot{\theta}^2).\vec{u}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}).\vec{u}_\theta$

$$= \gamma_\rho\vec{u}_\rho + \gamma_\theta\vec{u}_\theta$$

Its modulus: $\|\vec{\gamma}\| = \sqrt{\gamma_\rho^2 + \gamma_\theta^2} = \sqrt{(\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2}$

3.2.1.3 In Cylindrical coordinate system of the basis $(\vec{u}_\rho, \vec{u}_\theta, \vec{u}_z)$

- *Position Vector:* $\overrightarrow{OM} = \rho.\vec{u}_\rho + z.\vec{u}_z$

Its modulus: $\|\overrightarrow{OM}\| = \sqrt{\rho^2 + z^2}$

- *Velocity vector:* $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \dot{\rho}.\vec{u}_\rho + \rho.\dot{\theta}.\vec{u}_\theta + \dot{z}.\vec{u}_z$

$$= v_\rho.\vec{u}_\rho + v_\theta.\vec{u}_\theta + v_z.\vec{u}_z$$

Its modulus: $\|\vec{v}\| = \sqrt{v_\rho^2 + v_\theta^2 + v_z^2} = \sqrt{\dot{\rho}^2 + (\rho.\dot{\theta})^2 + \dot{z}^2}$

- *Acceleration vector:* $\vec{\gamma} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\overrightarrow{OM}}{dt^2}$

$$= (\ddot{\rho} - \rho\dot{\theta}^2).\vec{u}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}).\vec{u}_\theta + \ddot{z}.\vec{u}_z$$

$$= \gamma_\rho\,\vec{u}_\rho + \gamma_\theta\,\vec{u}_\theta + \gamma_z\,\vec{u}_z$$

Its modulus: $\|\vec{\gamma}\| = \sqrt{\gamma_\rho^2 + \gamma_\theta^2 + \gamma_z^2}$

$$= \sqrt{(\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2 + \ddot{z}^2}$$

3.2.1.4 In the Spherical coordinate system of the basis $(\vec{u}_r, \vec{u}_\varphi, \vec{u}_\theta)$

- *Position Vector:* $\overrightarrow{OM} = \vec{r} = r.\vec{u}_r$

Its modulus: $\|\overrightarrow{OM}\| = r$

- *Velocity Vector :* $\vec{v} = \dfrac{d\overrightarrow{OM}}{dt} = \dot{r}.\vec{u}_r + r.\sin\varphi.\dot{\theta}.\vec{u}_\theta + r.\dot{\varphi}.\vec{u}_\varphi$

$$= v_r\,.\vec{u}_r + v_\theta\,.\vec{u}_\theta + v_\varphi\,.\vec{u}_\varphi$$

Its modulus: $\|\vec{v}\| = \sqrt{v_r^2 + v_\theta^2 + v_\varphi^2} = \sqrt{\dot{r}^2 + (r.\sin\varphi.\dot{\theta})^2 + (r.\dot{\varphi})^2}$

- *Acceleration Vector:*

$$\vec{\gamma} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = (\ddot{r} - r.\dot{\varphi}^2 - r.\sin^2\varphi.\dot{\theta}^2)\,\vec{u}_r$$

$$+ (2r.\cos\varphi.\dot{\varphi}.\dot{\theta} + 2\dot{r}.\sin\varphi.\dot{\theta} + r.\sin\varphi.\ddot{\theta})\,\vec{u}_\theta$$

$$+ (2\dot{r}.\dot{\varphi} + r.\ddot{\varphi} + r.\sin\varphi.\cos\varphi.\dot{\theta}^2)\,\vec{u}_\varphi$$

$$= \gamma_r \vec{u}_r + \gamma_\theta \vec{u}_\theta + \gamma_\varphi \vec{u}_\varphi$$

Its modulus: $\|\vec{\gamma}\| = \sqrt{\gamma_r{}^2 + \gamma_\theta{}^2 + \gamma_\varphi{}^2}$

3.2.1.5 In intrinsic coordinates of the *Serret-Frenet* basis ($\vec{u}_T$, $\vec{u}_N$, $\vec{b}$)

For each point *M* of a curve *(C)*, it is possible to associate the trihedron with origin *M*, which is frame of reference tangent to the curve whose axes are defined by the unit vectors $\vec{u}_T$, $\vec{u}_N$ and $\vec{b}$, with :

$$\begin{cases} \vec{u}_T = \dfrac{d\overrightarrow{OM}}{dS} = \dfrac{\vec{v}}{v} \\ \dfrac{d\vec{u}_T}{dS} = \dfrac{\vec{u}_N}{R} \\ \vec{b} = \vec{u}_T \wedge \vec{u}_N \end{cases}$$

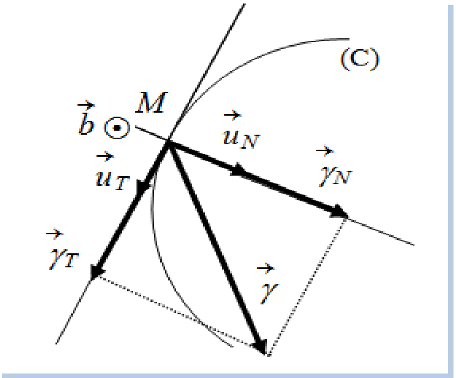


Figure 3.1 : Serret-Frenet basis

Where: dS is the elementary variation of the curvilinear abscissa, $\vec{u}_T$ is the unit vector of the tangent, *R* is the radius of curvature of the trajectory, $\vec{u}_N$ is the unit vector for the principal normal, $\vec{b}$ is the bi-normal (orthogonal to $\vec{u}_T$ and to $\vec{u}_N$).

The trihedron constructed on the direct basis ($\vec{\boldsymbol{u}}_{\boldsymbol{T}}$, $\vec{\boldsymbol{u}}_{N}$, $\vec{\boldsymbol{b}}$) is called the *Serret-Frenet* trihedron, and the plane containing the vectors $\vec{\boldsymbol{u}}_{\boldsymbol{T}}$ and $\vec{\boldsymbol{u}}_{\boldsymbol{N}}$ is called the osculating plane.

Curvilinear abscissa: $S(t) = S(M) = \overset{\frown}{M_0M}$; the curvilinear abscissa at time *t*, noted *S(t)*, of a material point *M*, is the length of the arc $\overset{\frown}{M_0M}$ of the trajectory of *M* counted from an origin M_0 at $t_0 = 0s$.

- *Velocity vector:* $\vec{v} = v.\vec{u}_T = \frac{dS}{dt}.\vec{u}_T$; the components of the velocity vector are $\vec{v}\left(\frac{dS}{dt}, 0, 0\right)$.

Its modulus: $\|\vec{v}\| = v = \frac{dS}{dt}$.

- *Acceleration vector*: $\vec{\gamma} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}.\vec{u}_T + \frac{v^2}{R}.\vec{u}_N = \vec{\gamma}_T + \vec{\gamma}_N$

Or: $\gamma_T = \frac{dv}{dt}$ and $\gamma_N = \frac{v^2}{R}$.

The components of the acceleration vector are $\vec{\gamma}\,(\gamma_T, \gamma_N, 0)$.

Its modulus: $\|\vec{\gamma}\| = \sqrt{\gamma_T{}^2 + \gamma_N{}^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{R}\right)^2}$

The acceleration vector $\vec{\gamma}$ is the sum of a tangential acceleration vector $\vec{\gamma}_T$ carried by the tangent and a normal acceleration vector $\vec{\gamma}_N$ carried by the principal normal.

3.3 Rectilinear movement

A motion is a rectilinear if its trajectory is a straight line.

3.3.1 *Uniform rectilinear movement* is the motion of a point body moving in a straight line at constant speed in the observer's frame of reference.

3.3.2 *Uniformly varied rectilinear movement* is the motion where the trajectory is a portion of a straight line, and speed is a function of time. The acceleration keeps the same direction, the same sense and same value.

	uniform rectilinear movement	*uniformly varied rectilinear movement*
Position	$x = v.t + x_0$	$x = \frac{1}{2}. \gamma.t^2 + v_0.t + x_0$
Speed	$v = v_0 = cte$	$v = \gamma.t + v_0$
Acceleration	$\gamma = 0$	$\gamma = \gamma_0 = cte$

3.4 Circular movement

A movement is circular if its trajectory is a circle.

We identify the point on the circle by the angle φ that the right *OM* makes with the axis *Ox*.

3.4.1 *Uniform circular movement* is the movement if the angle φ increases in a uniform manner (way).

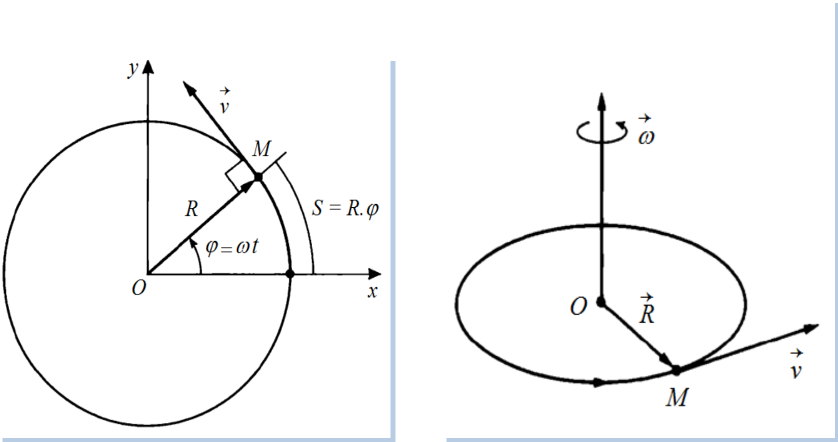


Figure 3.2: Circular movement

3.4.2 *Uniformly varied circular movement* is characterized by a circular trajectory and a constant angular acceleration.

	Uniform circular movement	*uniformly varied circular movement*
Position	$\varphi = \dot{\varphi}.t + \varphi_0$	$\varphi = \frac{1}{2}.\ddot{\varphi}.t^2 + \dot{\varphi}_0.t + \varphi_0$
Angular Speed	$\dot{\varphi} = \dot{\varphi}_0 = cte$	$\dot{\varphi} = \ddot{\varphi}.t + \dot{\varphi}_0$
Angular Acceleration	$\ddot{\varphi} = 0$	$\ddot{\varphi} = \ddot{\varphi}_0 = cte$

3.5 Kinematics with change of reference

3.5.1 Absolute and Relative Movements:

Let's consider a reference ($\mathscr{R}$) fixed over time, ($\mathscr{R}$') a moving reference (mobile) relative to ($\mathscr{R}$) and a point *M* that moves relative to ($\mathscr{R}$) and to ($\mathscr{R}$') (*Figure 3.3*).

The movement of a point *M* relative to ($\mathscr{R}$) is called *absolute movement*, and relative to ($\mathscr{R}$') is called *relative movement*.

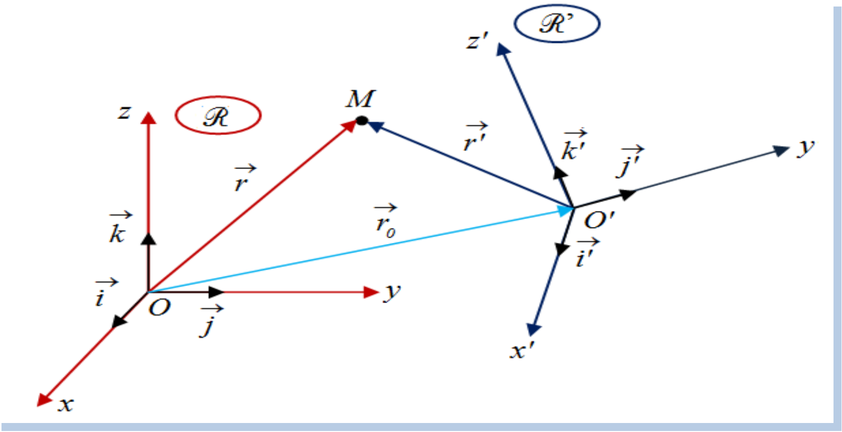


Figure 3.3: Absolute and Relative Movements:

3.5.2 Velocity composition theorem :

The absolute velocity vector is equal to the sum of the relative and training velocity vectors: $\vec{v}_a = \vec{v}_e + \vec{v}_r$.

Demonstration:

It is considered that *(x, y, z)* are the coordinates of the point *M* in the fixed reference ($\mathscr{R}$)$(O, \vec{i}, \vec{j}, \vec{k})$ and *(x', y', z')* are those of the point *M* in the moving reference ($\mathscr{R}$')$(O', \vec{i}', \vec{j}', \vec{k}')$.

From the *Figure 3.3*, we have: $\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$.

Or : $\overrightarrow{OM} = x.\vec{i} + y.\vec{j} + z.\vec{k}$; $\overrightarrow{O'M} = x'.\vec{i}' + y'.\vec{j}' + z'.\vec{k}'$.

The derivative of the vector $\overrightarrow{OM}$, gives:

$$\vec{v}_a = \frac{d\overrightarrow{OM}}{dt} = \frac{d\overrightarrow{OO'}}{dt} + x'.\frac{d\vec{i}'}{dt} + y'.\frac{d\vec{j}'}{dt} + z'.\frac{d\vec{k}'}{dt} + \frac{dx'}{dt}.\vec{i}' + \frac{dy'}{dt}.\vec{j}' + \frac{dz'}{dt}.\vec{k}'.$$

The term $\left(\frac{d\overrightarrow{OO'}}{dt} + x'.\frac{d\vec{i}'}{dt} + y'.\frac{d\vec{j}'}{dt} + z'.\frac{d\vec{k}'}{dt}\right)$ is the *training velocity* ($\vec{v}_e$) and represents the velocity of the moving reference ($\mathscr{R}'$) relative to the fixed reference ($\mathscr{R}$) (*velocity of the coincident point*).

The term $\left(\frac{dx'}{dt}.\vec{i}' + \frac{dy'}{dt}.\vec{j}' + \frac{dz'}{dt}.\vec{k}'\right)$ is the *relative velocity* ($\vec{v}_r$) which represents the velocity of the point M in the moving reference ($\mathscr{R}'$).

It can be demonstrate that:

$$\vec{v}_e = \frac{d\overrightarrow{OO'}}{dt} + \overset{\smile}{\omega}_e \wedge \overrightarrow{O'M}$$

3.5.3 Acceleration composition theorem :

The absolute acceleration vector is equal to the sum of the relative, training and Coriolis acceleration vectors:

$$\vec{\gamma}_a = \vec{\gamma}_r + \vec{\gamma}_e + \vec{\gamma}_c$$

Demonstration:

By definition, the absolute acceleration $\vec{\gamma}_a$ will be: $\vec{\gamma}_a = \frac{d\vec{v}_a}{dt}$.

If we derive the vector $\vec{v}_a$ with respect to time, we obtain the absolute acceleration vector defined in the fixed reference ($\mathscr{R}$):

$$\vec{\gamma}_a = \frac{d\,\vec{v}_a}{dt} = \frac{d^2\,\overrightarrow{OO'}}{dt^2} + \frac{d^2x'}{dt^2}\vec{i}\,' + \frac{d^2y'}{dt^2}\vec{j}\,' + \frac{d^2z'}{dt^2}\vec{k}\,' + 2.\frac{dx'}{dt}.\frac{d\vec{i}\,'}{dt} + 2.\frac{dy'}{dt}.\frac{d\vec{j}\,'}{dt}$$

$$+ 2.\frac{dz'}{dt}.\frac{d\vec{k}\,'}{dt} + x'\frac{d^2\vec{i}\,'}{dt^2} + y'\frac{d^2\vec{j}}{dt^2} + z'\frac{d^2\vec{k}\,'}{dt^2}$$

The term $\left(\frac{d^2\,\overrightarrow{OO'}}{dt^2} + x'\frac{d^2\vec{i}\,'}{dt^2} + y'\frac{d^2\vec{j}}{dt^2} + z'\frac{d^2\vec{k}\,'}{dt^2}\right)$ is called *training acceleration* $\vec{\gamma}_e$.

The term $\left(\frac{d^2x'}{dt^2}\vec{i}\,' + \frac{d^2y'}{dt^2}\vec{j}\,' + \frac{d^2z'}{dt^2}\vec{k}\,'\right)$ called *relative acceleration* $\vec{\gamma}_r$.

The term $\left(2.\left(\frac{dx'}{dt}.\frac{d\vec{i}\,'}{dt} + \frac{dy'}{dt}.\frac{d\vec{j}\,'}{dt} + \frac{dz'}{dt}.\frac{d\vec{k}\,'}{dt}\right)\right)$ *called Coriolis acceleration* $\vec{\gamma}_c$.

Finally, we find the acceleration composition theorem:

$$\vec{\gamma}_a = \vec{\gamma}_r + \vec{\gamma}_e + \vec{\gamma}_c.$$

It can be demonstrated that:

$$\vec{\gamma}_e = \frac{d^2\,\overrightarrow{OO'}}{dt^2} + \frac{d\vec{\omega}_e}{dt} \wedge \overrightarrow{O'M} + \vec{\omega}_e \wedge (\vec{\omega}_e \wedge \overrightarrow{O'M}).$$

$$\vec{\gamma}_c = 2.\,\vec{\omega}_e \wedge \vec{v}_r$$

Summaries of Exercises

Exercise 01

In an orthonormal Cartesian reference *(XOY)* with an orthonormal Cartesian basis $(O, \vec{i}, \vec{j})$, a material point *M* is identified by its Cartesian coordinates $M\ (at\ ;\ bt^2)$; where *t* represents the time in (*s*). *a* and *b* are constants.

1) Determine the physical meanings of the constants *a* and *b*.
2) Express the position, velocity and acceleration vectors of point *M*.
3) Find the equation of the trajectory of point *M*, and deduce its nature.

Exercise 02

In an orthonormal Cartesian reference *(Oxy)*, we consider a body *M* moving according to the following hourly equations: $\begin{cases} x = -t + 1 \\ y = \dfrac{1}{2}t - 2 \end{cases}$

Where: *x* and *y* are in meters and *t* is in seconds.

1) Find the trajectory equation of the body *M*.
2) Plot the trajectory in the following cases: $t \in [0\ ;\ 2]\ s$; $t \geq 0s$, and deduce its nature in both cases.
3) Determine the radius *R* of curvature.

Exercise 03

The curvilinear abscissa of a moving particle is given by the following relationship:

$$S(t) = t^3 + 2.t^2 (cm).$$

The modulus of its acceleration is: $\gamma = 16.\sqrt{2}\ (cm.s^{-2})$.

1) Find the moduli of the tangential and normal accelerations, at $t = 2s$.
2) Deduce the radius of curvature at time $t = 2s$.

Exercise 04

The movement of a material point *M*, moving the plane *(Oxy)* of the basis $(\vec{i}, \vec{j})$, is characterized by its position vector given by:

$$\overrightarrow{OM} = (2t + 2)\vec{i} + \left(\frac{1}{2}t^2 + t + \frac{1}{2}\right)\vec{j}$$

1) Give the trajectory equation of a point *M*. What is its nature?
2) Calculate the velocity of a point *M* and give its modulus.
3) Calculate the acceleration of a point *M* and give its modulus.
4) Determine the tangential and normal acceleration components of *M*.
5) Deduce the radius *R* of curvature from the trajectory at time $t=0s$.

Exercise 5

The coordinates of a particle *M* in the reference frame *(ℛ)* with a fixed reference $(O, \vec{i}, \vec{j}, \vec{k})$ are : $\begin{cases} x(t) = t^2 + 4t \\ y(t) = -2t^3 \\ z(t) = t^2 \end{cases}$ and in a reference frame *(ℛ')* with a mobile reference $(O', \vec{i}', \vec{j}', \vec{k}')$ with respect to *(ℛ)* are: $\begin{cases} x'(t) = t^2 - t + 2 \\ y'(t) = -2t^3 + 1 \\ z'(t) = t^2 - 1 \end{cases}$

It is assumed that: $\vec{i} = \vec{i}'$, $\vec{j} = \vec{j}'$ and $\vec{k} = \vec{k}'$.

1) Express the velocity vector $\vec{v}$ in *(ℛ)* and the velocity vector $\vec{v}'$ in *(ℛ')*.
2) Deduce the training velocity.
3) Express the acceleration $\vec{\gamma}$ in *(ℛ)* and the acceleration $\vec{\gamma}'$ in *(ℛ')*.
4) Deduce the training and Coriolis accelerations.

Exercise 6

A relative reference *R(O, xyz)* is considered to rotate uniformly around the absolute reference $R_0(O_0, x_0y_0z_0)$ so that their origins *O* and O_0 and their axes *Oz* and O_0z_0 remain coincident (confused) at all times *t*. Which $\theta = \omega t$ be the angle between the axes *Ox* and O_0x_0.

Consider a bar *(D)* fixed in the reference *R*, parallel to *Oz* axis and passing through the point *A*, where: $\overrightarrow{OA} = \alpha.\vec{j}$ (*Figure 3.6*)

A material point *M* moves on this bar according to the following relation:

$$\overrightarrow{AM} = e^{-\theta}.\vec{k}$$

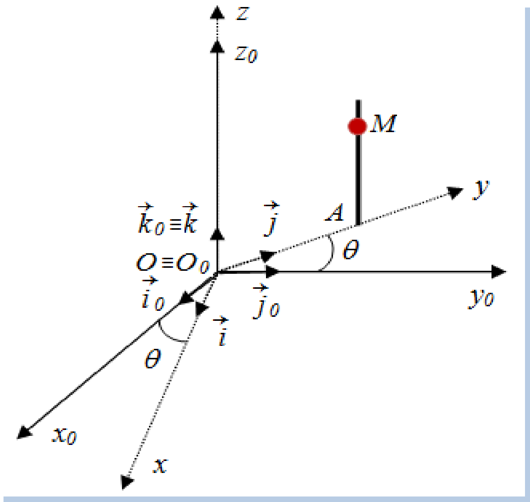


Figure 3.6

1) Determine the relative velocity and acceleration of point *M*.
2) Find the training (drive) velocity and acceleration.
3) Calculate the Coriolis acceleration.
4) Deduce the absolute velocity and acceleration of the point *M*.

Exercises' Solutions

Exercise 01

We have the hourly equations of motion: $\begin{cases} x = a.t \\ y = b.t^2 \end{cases}$

1/ Determining the physical meaning of the constants *a* and *b*:

Using dimensional analysis, we obtain:

$[x] = [a].[t]$ therefore : $[a] = \dfrac{[x]}{[t]} = L.T^{-1}$; then : a represents the velocity.

$[y] = [b].[t]^2$ therefore : $[b] = \dfrac{[y]}{[t]^2} = L.T^{-2}$; so : b represents the acceleration.

2/ Position, velocity and acceleration vectors:

The position vector $\overrightarrow{OM}$: $\overrightarrow{OM} = x\,\vec{i} + y\,\vec{j} = (a.t).\vec{i} + (b.t^2).\vec{j}$

The velocity vector $\vec{v}$: $\vec{v} = \dfrac{d\overrightarrow{OM}}{dt} = (a).\vec{i} + (2.b.t).\vec{j}$

The acceleration vector $\vec{\gamma}$: $\vec{\gamma} = \dfrac{d\vec{v}}{dt} = (2.b).\vec{j}$

3/ Equation of the trajectory:

We have: $\begin{cases} x = a.t & (1) \\ y = b.t^2 & (2) \end{cases}$

From the relation (1) we have: $t = \dfrac{x}{a}$, replacing in (2), we get the equation of the following trajectory: $y = \dfrac{b}{a^2}\,.x^2 \qquad (3)$

The last equation (3) is an equation of a parable. So, the trajectory is a parabolic arc.

Exercise 02

We have the following hourly equations: $\begin{cases} x = -t + 1 & (1) \\ y = \dfrac{1}{2}t - 2 & (2) \end{cases}$

1/ Trajectory equation:

According to the relation (1), we have: $t = -x + 1$, replacing in (2), we get the equation of the trajectory: $y = -\dfrac{1}{2}x - \dfrac{3}{2}$.

2/ Representing the trajectory in the following cases: $t \in [0\ ;\ 2]$ s ; $t \geq 0s$.

1st case: **t ∈ [0 ; 2] s**

When: $t = 0\ s$ *, we find :* $x = 1$ *and* $y = -2$

$t = 2\ s$ *, we find :* $x = -1$ *and* $y = -1$

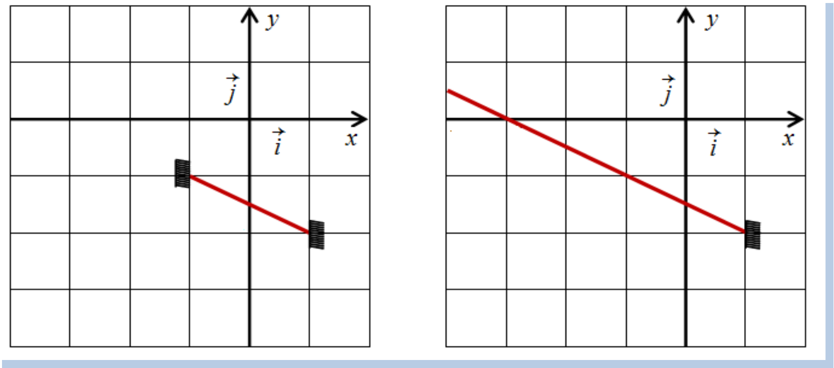
The nature of the trajectory is a segment of a straight line.

2nd case: **t ≥ 0 s**

When: $t = 0\ s$ *, we find :* $x = 1$ *and* $y = -2$

$t \rightarrow +\infty$ *, we find :* $x \rightarrow -\infty$ *and* $y \rightarrow +\infty$

The nature of the trajectory is semi-straight line.



1st case *2nd case*

Figure 3.9

3/ Determining the radius of curvature:

First, we find the components of velocity and acceleration:

The position vector : $\overrightarrow{OM} = x\,\vec{i} + y\,\vec{j} = (-t+1).\vec{i} + (\frac{1}{2}t - 2).\vec{j}$

The velocity vector : $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = -\vec{i} + \frac{1}{2}.\vec{j}$

The acceleration vector : $\vec{\gamma} = \frac{d\vec{v}}{dt} = \vec{0}$

We now calculate the tangential and normal acceleration:

By definition: $\gamma_T = \frac{d\|\vec{v}\|}{dt}$; as the modulus of $\vec{v}$ is:

$\|\vec{v}\| = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$; therefore : $\gamma_T = 0$

As well as: $\gamma^2 = \gamma_T^2 + \gamma_n^2$ so: $\gamma_n^2 = \gamma^2 - \gamma_T^2$, we obtain : $\gamma_N = 0$

The curvature radius is defined by: $R_C = \frac{v^2}{\gamma_N}$

Then: $R_C = \frac{v^2}{\gamma_n} = +\infty$.

Exercise 03

The curvilinear abscissa of a particle is given by: $S(t) = t^3 + 2.t^2\ (cm)$.

The modulus of its acceleration is: $\gamma = 16.\sqrt{2}\,(cm.s^{-2})$.

1/ Tangential and normal acceleration at the instant $t = 2s$

Modulus of the velocity : $v = \frac{dS}{dt} = 3.t^2 + 4.t$

Implies: at the instant $t = 2s$: $v(2s) = 20\ cm/s$

Tangential acceleration: $\gamma_T = \frac{dv}{dt} = 6.t + 4$

Implies: at the instant $t = 2s$: $\gamma_T(2s) = 16\ m/s^2$

Normal acceleration: $\gamma_N(2s) = \sqrt{\gamma^2 - \gamma_T^2} = 16\ cm/s^2$

2/ Deduction of the R_C at the instant $t = 2s$.

The radius of curvature is defined by: $R_C = \dfrac{v^2}{\gamma_N}$

$$So: R_C = \frac{v^2}{\gamma_n} = \frac{20^2}{16} = 25\ cm.$$

Exercise 04

The parametric equations of the material point M are:

$$\overrightarrow{OM} = \begin{cases} x = 2.t + 2 & (1) \\ y = \dfrac{1}{2}.t^2 + t + \dfrac{1}{2} & (2) \end{cases}$$

1/ Equation of the trajectory of the material point *M* :

To determine the equation of the trajectory, we eliminate the time t :

So, according to the equation (1) we have: $t = \dfrac{x-2}{2}$ (3)

By replacing equation (3) in equation (2), we obtain:

$$y = \frac{1}{2}.\frac{(x-2)^2}{4} + \frac{x-2}{2} + \frac{1}{2} = \frac{1}{8}.x^2 \qquad (4)$$

The nature of the trajectory:

The equation (4) is an equation of a parabola, so the trajectory is a parabolic arc.

2/ Calculation of the velocity of the material point *M* and its modulus:

The velocity vector is by definition:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = 2.\vec{i} + (t+1).\vec{j}$$

Its modulus: $\|\vec{v}\| = \sqrt{4 + (t+1)^2} = \sqrt{t^2 + 2.t + 5}\ m/s$

Exercise 5

The mobile *M* is identified in the absolute reference $(\mathscr{R})$ by: $\begin{cases} x(t) = t^2 + 4t \\ y(t) = -2t^3 \\ z(t) = t^2 \end{cases}$

And identified in the relative reference $(\mathscr{R}')$ by: $\begin{cases} x'(t) = t^2 - t + 2 \\ y'(t) = -2t^3 + 1 \\ z'(t) = t^2 - 1 \end{cases}$

1/ Velocity vector $\vec{v}$ in $(\mathscr{R})$ and the velocity vector $\vec{v}'$ in $(\mathscr{R}')$:

$$\vec{v}\Big|_{(\mathscr{R})} = \begin{cases} v_x = \dfrac{dx}{dt} = 2.t + 4 \\ v_y = \dfrac{dy}{dt} = -6.t^2 \\ v_z = \dfrac{dz}{dt} = 2.t \end{cases}$$

$$\vec{v}'\Big|_{(\mathscr{R}')} = \begin{cases} v'_x = \dfrac{dx'}{dt} = 2.t - 1 \\ v'_y = \dfrac{dy'}{dt} = -6.t^2 \\ v'_z = \dfrac{dz'}{dt} = 2.t \end{cases}$$

2/ Training (drive) velocity $\vec{v}_e$:

We have: $\begin{cases} v_x = v'_x + 5 \\ v_y = v'_y \\ v_z = v'_z \end{cases}$

According to the velocity composition law: $\vec{v}_a = \vec{v}_r + \vec{v}_e$

We find that: $\vec{v}_e = \vec{v}_a - \vec{v}_r = 5.\vec{i}$

Conclusion: The relative reference $(\mathscr{R}')$ moves at a constant speed of $5\ m/s$ about an absolute reference $(\mathscr{R})$.

The relative reference $(\mathscr{R}')$ is following the unit vector $\vec{i}$ of the axis (Ox).

3/ Expression of acceleration vector $\vec{\gamma}$ in the reference $(\mathscr{R})$ and accelerating vector $\vec{\gamma}'$ in the reference $(\mathscr{R}')$:

$$\vec{\gamma}\Big|_{(\mathscr{R})} = \begin{cases} \gamma_x = \dfrac{dv_x}{dt} = 2 \\ \gamma_y = \dfrac{dv_y}{dt} = -12.t \\ \gamma_z = \dfrac{dv_z}{dt} = 2 \end{cases}$$

$$\vec{\gamma}'\Big|_{(\mathscr{R}')} = \begin{cases} \gamma'_x = \dfrac{dv'_x}{dt} = 2 \\ \gamma'_y = \dfrac{dv'_y}{dt} = -12.t \\ \gamma'_z = \dfrac{dv'_z}{dt} = 2 \end{cases}$$

So, we notice that: $\vec{\gamma}\Big|_{(\mathscr{R})} = \vec{\gamma}'\Big|_{(\mathscr{R}')}$

4/ Accelerations of training (drive) and Coriolis:

Training (drive) acceleration: $\vec{\gamma}_e = \dfrac{d\vec{v}_e}{dt} = \vec{0}$

To deduce the acceleration of Coriolis $\vec{\gamma}_c$, the law of composition of accelerations is applied:

$$\vec{\gamma}_a = \vec{\gamma}_r + \vec{\gamma}_e + \vec{\gamma}_c$$

So: The Coriolis acceleration: $\vec{\gamma}_c = \vec{\gamma}_a - \vec{\gamma}_r + \vec{\gamma}_e = \vec{0}$

Exercise 6

1/ Relative velocity and acceleration:

We have: $\overrightarrow{OA} = \alpha.\vec{j}$ and $\overrightarrow{AM} = e^{-\theta}.\vec{k}$

The position vector in the relative reference:

$$\overrightarrow{OM}\,(\mathscr{R}) = \overrightarrow{OA} + \overrightarrow{AM} = \alpha.\,\vec{j} + e^{-\theta}.\,\vec{k}$$

The relative velocity vector: $\vec{v}_r = \left.\dfrac{d\overrightarrow{OM}}{dt}\right|(\mathscr{R}) = -\,\omega.e^{-\theta}.\,\vec{k}$

The relative acceleration vector: $\vec{\gamma}_r = \left.\dfrac{d^2\overrightarrow{OM}}{dt^2}\right|(\mathscr{R}) = \omega^2.e^{-\theta}.\,\vec{k}$

2/ Training (driving) velocity and acceleration:

The training velocity vector:

$$\vec{v}_e = \frac{d\overrightarrow{O_0O}}{dt} + \vec{\omega} \wedge \overrightarrow{OM} = \vec{O} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega \\ 0 & \alpha & e^{-\theta} \end{vmatrix} = -\,\omega.\alpha.\vec{i}$$

The training acceleration vector:

$$\vec{\gamma}_e = \frac{d^2\overrightarrow{O_0O}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{OM} + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{OM})$$

$$\vec{\gamma}_e = \vec{O} + \vec{O} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega \\ -\,\omega.\alpha & 0 & 0 \end{vmatrix} = -\,\omega^2.\alpha.\,\vec{j}$$

3/ Coriolis acceleration:

$$\vec{\gamma}_c = 2.(\vec{\omega} \wedge \vec{v}_r) = 2.\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \omega \\ 0 & 0 & -\,\omega.e^{-\theta} \end{vmatrix} = \vec{0}$$

4/ Absolute velocity and acceleration:

We have : $\vec{v}_a = \vec{v}_r + \vec{v}_e$ and $\vec{\gamma}_a = \vec{\gamma}_r + \vec{\gamma}_e + \vec{\gamma}_c$

So, in the relative reference $R(O, x, y, z)$, we obtain :

$$\vec{v}_a = \vec{v}_r + \vec{v}_e = -\omega.\alpha.\vec{i} - \omega.e^{-\theta}.\vec{k}$$

$$\vec{\gamma}_a = \vec{\gamma}_r + \vec{\gamma}_c + \vec{\gamma}_e = -\omega^2.\alpha.\vec{j} + \omega^2.e^{-\theta}.\vec{k}$$

In the absolute reference $R_0(O_0, x_0, y_0, z_0)$, we have:

$$\vec{i} = (\cos\theta.\vec{i}_0 + \sin\theta.\vec{j}_0)\ ,\ \vec{j} = (-\sin\theta.\vec{i}_0 + \cos\theta.\vec{j}_0) \text{ and } \vec{k} = \vec{k}_0$$

Therefore:

$$\vec{v}_a = -\omega.\alpha(\cos\theta.\vec{i}_0 + \sin\theta.\vec{j}_0) - \omega.e^{-\theta}.\vec{k}_0$$

$$\vec{\gamma}_a = -\omega^2.\alpha.(-\sin\theta.\vec{i}_0 + \cos\theta.\vec{j}_0) + \omega^2.e^{-\theta}.\vec{k}_0$$

Chapter	Dynamics of the Material Point

4. Dynamics of the Material Point

4.1 Definition:

Dynamics is a discipline of classical mechanics. It is the study of the causes, which provoke the movements of solid bodies *(the forces and the actions applied).*

4.2 Galilean or Inertia Reference

A Galilean or inertia reference is a referential for which a material point of mass m is isolated; either in rest or in uniform rectilinear motion.
Any other reference at rest, or in uniform rectilinear motion in relation to an inertia reference, is identical to an Inertia Reference.

4.3 Newton's laws:

Newton's laws are the basis of classical mechanics. They were postulated without demonstration, but they are in good agreement with the experiments.

The statements of *Newton's* three laws are as follows:

- ***Newton's first law or principle of inertia:***

  In a Galilean reference, a material point remains immobile or in uniform rectilinear motion, when the resulting force exercising on it is zero (null): $\sum \vec{F}_{ext} = \vec{0} \Rightarrow \vec{\gamma} = \vec{0} \Rightarrow \vec{v} = \overrightarrow{cte}$.

- ***Newton's second law or fundamental principle of dynamics:***

  In a Galilean reference, the variation in the amount of motion of a material point relative to time is equal to the resulting force applied to that material point: $\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$

  This relation is also written: $\sum \vec{F}_{ext} = m.\vec{\gamma}$; if the mass of the material point is constant.

- ***Newton's third law or principle of mutual actions:***

  Any body *A* exerting a force on body *B* is subjected to a force of equal intensity, of the same direction and of opposite direction, exerted by body *B*. So : $\vec{F}_{AB} = -\vec{F}_{BA}$.

  This law is sometimes called the principle of action – reaction.

4.4 Quantity of motion:

In a Galilean reference, we consider a material point *M* of mass *m* animated by the velocity vector $\vec{v}$. The amount of motion of the point *M* is the $\vec{P}$ vector defined by the relation:

$$\vec{P} = m.\vec{v}$$

According to the fundamental principle of dynamics (*F.P.D*), we have:

$$\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

Where: $\vec{F} = \sum \vec{F}_{ext} = \frac{d\vec{P}}{dt} = \frac{d(m.\vec{v})}{dt} = m.\frac{d\vec{v}}{dt} + \frac{dm}{dt}.\vec{v}$

We can write this expression in the form: $\vec{F} = \frac{d\vec{P}}{dt} = m.\frac{d\vec{v}}{dt} = m.\vec{\gamma}$, if the mass of the material point is constant during movement $\left(\frac{dm}{dt} = 0\right)$.

4.5 Types of forces in nature

4.5.1 Remote forces:

Remote forces can be exercised without bodies being in contact. Among which one can cite:

- *Electrostatic force:* $\vec{F} = k.\frac{q_1.q_2}{r^2}.\vec{u} = \frac{1}{4\pi\varepsilon_0}.\frac{q_1.q_2}{r^2}.\vec{u}$; *where:* q_1 and q_2 are the electrical charges of the two particles, r is the distance between the charges, ε_0 is the permittivity of the vacuum, $k = \frac{1}{4\pi\varepsilon_0} = 9.10^9\ N.m^2.C^{-2}$ is the Coulomb constant, $\vec{u}$ is the unit vector of the force $\vec{F}$.
- *Magnetic force:* $\vec{F}_m = q.\vec{v} \wedge \vec{B}$. *where:* q is the electrical charge of the moving particle, $\vec{v}$ is the velocity of the particle and, $\vec{B}$ is the magnetic field.
- *Electromagnetic force:* $\vec{F} = q.(\vec{E} + \vec{v} \wedge \vec{B})$. *where:* q is the electrical charge of the moving particle, $\vec{v}$ is the velocity of the particle, $\vec{E}$ is the

electric field, $\vec{B}$ is the magnetic field.

- *Universal gravitational (attraction) force:* $\vec{F} = G.\frac{m_1.m_2}{r^2}.\vec{u}$, *where:* m_1 and m_2 are masses of the two particles, r is the distance between masses, $G = 6.67.\ 10^{-11}\ N.m^2.kg^{-2}$, $\vec{u}$ is the unit vector of the force $\vec{F}$.

4.5.2 Contact forces:

These forces require contact. They occur when one body is in contact with another body. Among these forces, we distinguish mainly:

- Mechanical constraints.
- Forces of friction.
- The cohesion forces of matter.
- Chemical bonds.
- Nuclear interactions.

4.6 Kinetic moment

4.6.1 Moment of a force about a fixed point

The moment of a force $\vec{F}$ about a fixed point A is defined by:

$$\vec{M}_A(\vec{F}) = \overrightarrow{AB} \wedge \vec{F}$$

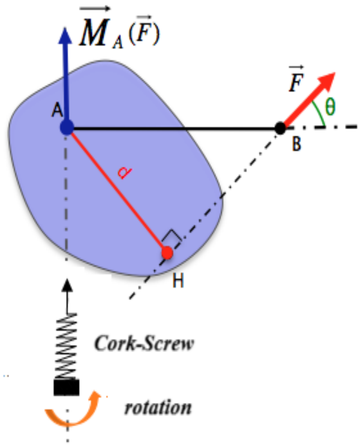


Figure 4.1: Moment of a force about a point

Where: B is any point in the line of action of the force $\vec{F}$.

By the properties of the vector product, the moment vector is perpendicular to both the force $\vec{F}$ and the vector $\overrightarrow{AB}$. Its orientation is given by the *Cork-Screw* rule.

4.6.2 Moment of a force about an axis

The moment of a force $\vec{F}$ about an oriented axis (Δ) is defined by:

$$\vec{M}_u(\vec{F}) = \vec{u}.\vec{M}_A(\vec{F}) = \vec{u}.\,(\overrightarrow{AB} \wedge \vec{F})$$

Where $\vec{u}$ is the unit vector of the axis (Δ).

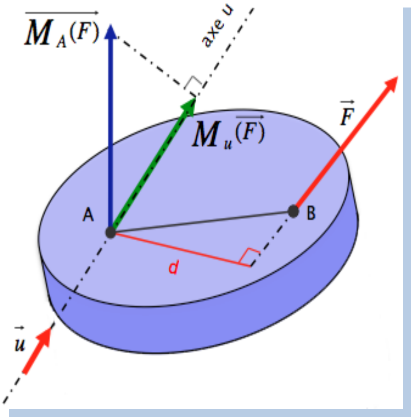


Figure 4.2: Moment of a force about an axis

The moment of a force $\vec{F}$ about an axis is therefore a *scalar* magnitude.

4.6.3 Kinetic moment about a fixed point

The kinetic moment of a material point *M*, about a fixed point *A* of space marked by $\vec{L}_A$, is the moment of its quantity of motion $\vec{p}$. It is given by the following relation:

$$\vec{L}_A = \overrightarrow{AM} \wedge \vec{p}$$

4.6.4 Kinetic moment theorem

In a Galilean reference, the derivative with respect to time of the kinetic moment about a fixed point *A*, is equal to the moment about a point *A* of the resultant of the external forces applied to the point *M*.

$$\frac{d\vec{L}_A}{dt} = \vec{M}_A\left(\sum \vec{F}_{ext}\right)$$

Demonstration:

$$\frac{d\vec{L}_A}{dt} = \frac{d(\overrightarrow{AM} \wedge \vec{p})}{dt} = \left(\frac{d\overrightarrow{AM}}{dt} \wedge \vec{p}\right) + \left(\overrightarrow{AM} \wedge \frac{d\vec{p}}{dt}\right)$$

$$= \left(\frac{d\overrightarrow{OM}}{dt} \wedge \vec{p}\right) - \left(\frac{d\overrightarrow{OA}}{dt} \wedge \vec{p}\right) + \left(\overrightarrow{AM} \wedge \frac{d\vec{p}}{dt}\right)$$

Knowing that: $\left(\frac{d\overrightarrow{OM}}{dt} \wedge \vec{p}\right) = \vec{v} \wedge m.\vec{v} = \vec{0}$

The points O and A are fixed, $\overrightarrow{OA}$ is a constant vector, its derivative about a time is zero (null), so: $\left(\frac{d\overrightarrow{OA}}{dt} \wedge \vec{p}\right) = \vec{0}.$

According to the quantity of motion theorem:

$\left(\overrightarrow{AM} \wedge \frac{d\vec{p}}{dt}\right) = \overrightarrow{AM} \wedge \sum \vec{F}_{ext}$; so : $\frac{d\vec{L}_A}{dt} = \frac{d(\overrightarrow{AM} \wedge \vec{p})}{dt} = \vec{M}_A(\sum \vec{F}_{ext})$

Notes:

- If: $\vec{M}_A(\sum \vec{F}_{ext}) = \overrightarrow{cte}$ $\Rightarrow$ The motion is a central acceleration plane and the point A is the center of the accelerations.
- If: $\vec{M}_A(\sum \vec{F}_{ext}) = \vec{0}$ $\Rightarrow$ The motion is rectilinear, the trajectory of the material point is a straight.

Summaries of Exercises

Exercise 01

In the plane *(Oxy)* of the Galilean point *R(O, xyz)*, we are interested in the movement of a particle *A*, of mass *m*, subject to the action of the force $\vec{F}$. The movement is described in Polar coordinates $(\rho$ *and* $\theta)$ and using the Cylindrical basis $(\vec{u}_\rho, \vec{u}_\theta, \vec{u}_z)$.

1) Determine the components of the $\vec{\gamma}(A)_{/R}$ acceleration based on ρ, θ and their derivatives.
2) Show that: $\rho^2.\dot{\theta} = C$ (considering *C*, constant of areas).
3) a– Express the kinetic moment of *A* at point *O*: $\vec{M}_O(A)_{/R}$.

   b– Show that this kinetic moment $\vec{M}_O(A)_{/R}$ is preserved and deduce from it that its modulus can be written: $\|\vec{M}_O(A)_{/R}\| = m.\rho^2.\dot{\theta} = m.C$

Exercise 02

A particle *M* of mass *m* is moving in a plane (*xOy*). This particle is subject to two forces $\vec{F}_1 = m.\vec{v}$ and $\vec{F}_2 = -2.\overrightarrow{OM}$.

1/ In the Polar coordinates system of the basis $(\vec{u}_\rho, \vec{u}_\theta)$.

- Find differential equations of the movement of the particle.
- Given that: at the instant $t = 0$: $\rho = A$, and assuming that : $\theta = \omega t$ where : ω and A are constants. Show that the hourly equation of motion is given by: $\rho = A.e^{2t}$

2/ Using the kinetic moment theorem :

– Find the hourly equation of motion: $\rho = A.e^{2t}$.

Exercise 03

A mobile *M* of a mass $m = 150\ g$, can slide on a curvilinear (semi-circle) trajectory, the shape of which is shown in *Figure 4.3* ; the movement takes place in a vertical plane. The mobile *M* is launched in *A* with zero (null) velocity $v_A = 0\ m/s$ and directed downward. It slides without friction or motor action.

We give : the radius of the semi-circle $R = 1{,}36\ m$, $g = 10\ m/s^2$.

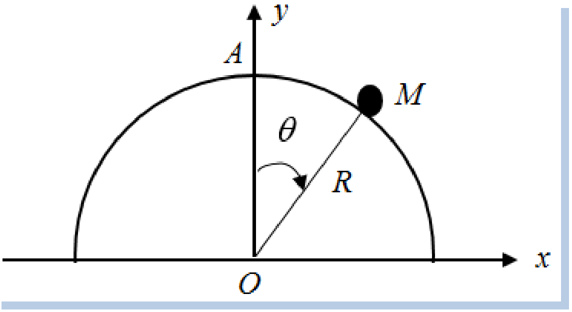


Figure 4.3

Answer the following questions using the following two methods:

I– The fundamental principle of dynamics (*F.P.D*)

II– The kinetic moment theorem (*K.M.T*)

a) Study the movement of the mobile in *the Serret-Frenet* basis.

b) Find the expression of velocity, as well as the reaction force of the trajectory.

c) Find the position where the mobile leaves the circular trajectory.

d) Then calculate the speed of the mobile at this position.

Exercises' Solutions

Exercise 01

1/ Components of the acceleration $\vec{\gamma}(A)_{/R}$:

In basic Polar coordinates system $(\vec{u}_\rho, \vec{u}_\theta)$:

We have: $(\overrightarrow{OA})_{/R} = \rho.\vec{u}_\rho$ *(the position vector)*

And : $\vec{v}(A)_{/R} = \dfrac{d\overrightarrow{OA}}{dt} = \dot{\rho}.\vec{u}_\rho + \rho.\dot{\theta}.\vec{u}_\theta$ *(the velocity vector).*

Which produces the acceleration vector:

$$\vec{\gamma}(A)_{/R} = (\ddot{\rho} - \rho\dot{\theta}^2).\vec{u}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}).\vec{u}_\theta$$

2/ Showing that: $\rho^2.\dot{\theta} = C$:

The movement is central, so: $\vec{F} = F(\rho).\vec{u}_\rho$

By applying the fundamental principle of dynamics (*F.P.D*) to point *A*, we will have: $m.\vec{\gamma}(A)_{/R} = \vec{F}(\rho) = F(\rho).\vec{u}_\rho$

Which implies that: $(\ddot{\rho} - \rho\dot{\theta}^2) = F(\rho)$ *(1)*

$$(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}) = 0 \qquad (2)$$

(2) implies: $\dfrac{1}{\rho}.\dfrac{d}{dt}(\rho^2\dot{\theta}) = 0 \Rightarrow \rho^2\dot{\theta} = Cte = C$

3/ a– Kinetic moment from *A* to point *O*: $\vec{M}_O(A)_{/R}$:

The kinetic moment $\vec{M}_O(A)_{/R} = \overrightarrow{OA} \wedge m.\vec{v}(A)_{/R} = m.\rho^2\dot{\theta}.\vec{u}_z$

b/ Showing that the kinetic moment $\vec{M}_O(A)_{/R}$ is conserved:

On the other hand, as:

$$\frac{d}{dt}\left(\vec{M}_O(A)_{/R}\right) = \overrightarrow{OA} \wedge m.\vec{\gamma}(A)_{/R} = \rho.\vec{u}_\rho \wedge F(\rho).\vec{u}_\rho = \vec{0}$$

We find that: $\vec{M}_O(A)_{/R} = \overrightarrow{Cte}$

Then, the kinetic moment is conserved (constant).

Having: $\|\vec{M}_O(A)_{/R}\| = m.\rho^2.\dot{\theta}$

It is further inferred that: $\rho^2.\dot{\theta} = \dfrac{\|\vec{M}_O(A)\|}{m} = Cte = C.$

Exercise 02

A particle M subjected to two forces $\vec{F}_1 = m.\vec{v}$ and $\vec{F}_2 = -2.\overrightarrow{OM}$.

1/ Differential equations of particle motion:

Using the fundamental principle of dynamics:

$$\sum \vec{F}_{ext} = m\vec{\gamma} \ ; \text{ So: } \vec{F}_1 + \vec{F}_2 = m\vec{\gamma} \qquad (1)$$

We can write the two forces $\vec{F}_1$, $\vec{F}_2$ and the acceleration $\vec{\gamma}$, in the Polar basis $(\vec{u}_\rho, \vec{u}_\theta)$:

$$\vec{F}_1 = m(\dot{\rho}.\vec{u}_\rho + \rho.\dot{\theta}.\vec{u}_\theta) \qquad (2)$$

$$\vec{F}_2 = -2.\rho.\vec{u}_\rho \qquad (3)$$

$$\vec{\gamma} = (\ddot{\rho} - \rho\dot{\theta}^2)\,\vec{u}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\,\vec{u}_\theta \qquad (4)$$

By replacing equations (2), (3) and (4) in equation (1), we obtain:

$$(m.\dot{\rho} - 2\rho)\vec{u}_\rho + m.\rho.\dot{\theta}\,\vec{u}_\theta = (\ddot{\rho} - \rho\dot{\theta}^2)\,\vec{u}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\,\vec{u}_\theta$$

So, we have two equations: radial and ortho-radial, which are:

$$\begin{cases} m.\dot{\rho} - 2\rho = m(\ddot{\rho} - \rho\dot{\theta}^2) & \textit{(the radial equation)} \\ m.\rho.\dot{\theta} = m(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}) & \textit{(the ortho-radial equation)} \end{cases}$$

Knowing that: $\theta = \omega t \Rightarrow \dot{\theta} = \omega \Rightarrow \ddot{\theta} = 0.$

We will find the differential equations of the motion of the particle M:

$$\begin{cases} m.\dot{\rho} - 2\rho = m(\ddot{\rho} - \rho\dot{\theta}^2) & (5) \\ \rho = 2\dot{\rho} & (6) \end{cases}$$

Showing that the hourly equation of motion is given by:

$\boldsymbol{\rho = A.e^{2t}}$ **:**

To find the hourly equation of motion, we look for the solution of the differential equation (6).

We have: $\rho = 2\dot{\rho} \Rightarrow 2.\frac{d\rho}{dt} = \rho \Rightarrow \frac{d\rho}{\rho} = 2.dt$ $\qquad (7)$

We integrate the equation (7) :

$$\int \frac{d\rho}{\rho} = 2\int dt \Rightarrow ln\rho = 2t + lnk \Rightarrow \rho = k.e^{2t}$$

Based on the initial conditions: at the instant $t = 0$: $\rho = A \Rightarrow k = A$

We will find the hourly equation of the movement: $\rho = A.e^{2t}$.

2/ Hourly equation of movement $\boldsymbol{\rho = A.e^{2t}}$, using the kinetic moment theorem:

By definition, the kinetic moment theorem is :

$$\frac{d\vec{L}_0}{dt} = \vec{M}_O(\sum \vec{F}_{ext}) = \sum \overrightarrow{OM} \wedge \vec{F}_{ext}$$

Also, we have the relation of the kinetic moment : $\vec{L}_0 = \overrightarrow{OM} \wedge m\vec{v}$

Then : $\vec{L}_0 = \rho.\vec{u}_\rho \wedge m(\dot{\rho}.\vec{u}_\rho + \rho\dot{\theta}.\vec{u}_\theta) = m.\rho^2.\dot{\theta}.\vec{u}_z$

so: $\dfrac{d\vec{L}_0}{dt} = 2.m.\omega.\rho.\dfrac{d\rho}{dt}.\vec{u}_z$ (8)

We have two forces exerted on the particle, so we have two moments of force:

$$\overrightarrow{OM} \wedge \vec{F}_1 = \rho.\vec{u}_\rho \wedge \left(m(\dot{\rho}.\vec{u}_\rho + \rho\dot{\theta}.\vec{u}_\theta)\right) = m\omega\rho^2.\vec{u}_z \quad (9)$$

$$\overrightarrow{OM} \wedge \vec{F}_2 = \rho.\vec{u}_\rho \wedge (2\rho.\vec{u}_\rho) = \vec{0} \quad (10)$$

If equation (8) = equation (9), then:

$$2.m.\omega.\rho.\frac{d\rho}{dt}.\vec{u}_z = m\omega\rho^2.\vec{u}_z \Rightarrow 2.\rho = \frac{d\rho}{dt} \Rightarrow \frac{d\rho}{\rho} = 2.dt$$

By integrating the previous relation, we find:

$$\int \frac{d\rho}{\rho} = 2\int dt \Rightarrow ln\rho = 2.t + lnk \Rightarrow \rho = k.e^{2t} \Rightarrow \rho = A.e^{2t}$$

So, we got the same previous result.

Exercise 03

I/ Using the fundamental principle of dynamics:

a/ Study of movement of mobile *M*:

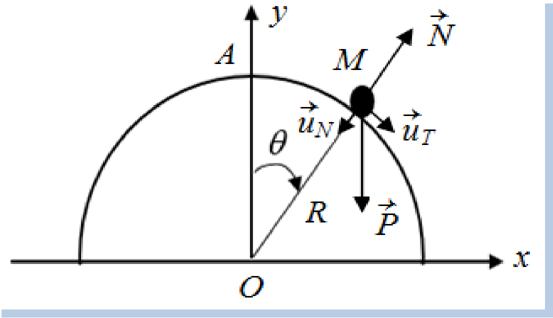


Figure 4.10

The mobile M of mass m is subject to two forces: its weight $\vec{P}$ and the reaction of the support $\vec{N}$.

The basic principle of dynamics is applied:

$$\sum \vec{F}_{ext} = m\vec{\gamma} \quad \Rightarrow \vec{P} + \vec{N} = m\vec{\gamma} \qquad (1)$$

The forces $\vec{P}$, $\vec{N}$ and the acceleration $\vec{\gamma}$, can be written in the intrinsic coordinates system (*Serret-Frenet basis*) (*see Figure 4.10*) , as follows:

$$\begin{cases} \vec{P} = mg.\sin\theta.\vec{u}_T + mg.\cos\theta.\vec{u}_N \\ \vec{N} = -N.\vec{u}_N \\ \vec{\gamma} = \dfrac{dv}{dt}.\vec{u}_T + \dfrac{v^2}{R}.\vec{u}_N \end{cases}$$

By replacing the expressions of $\vec{P}$, $\vec{N}$ and $\vec{\gamma}$ in the relation (1), we obtain:

$$mg.\sin\theta.\vec{u}_T + (mg.\cos\theta - N).\vec{u}_N = m.\frac{dv}{dt}.\vec{u}_T + m.\frac{v^2}{R}.\vec{u}_N \qquad (2)$$

According to the relation (2), we have two equations:

The tangential equation: $mg.\sin\theta = m.\dfrac{dv}{dt}$

$$\Rightarrow \quad \frac{dv}{dt} = g.\sin\theta \qquad (3)$$

The normal equation: $mg.\cos\theta - N = m.\dfrac{v^2}{R}$

$$\Rightarrow \quad g.\cos\theta - \frac{N}{m} = \frac{v^2}{R} \qquad (4)$$

b/ Velocity expression:

According to the tangential equation (3), we have: $\dfrac{dv}{dt} = g.\sin\theta$

$$\Rightarrow \frac{dv}{d\theta}\frac{d\theta}{dt} = g.sin\theta \;;$$

Where: $\frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R}$

So: $\frac{v}{R}.\frac{dv}{d\theta} = g.sin\theta$

$$\Rightarrow v.dv = R.g.sin\theta.d\theta \qquad (5)$$

Integrating the relation (5) :

$$\int_0^v v.dv = R.g.\int_0^\theta sin\theta.d\theta \Rightarrow \frac{v^2}{2} = R.g.[-cos\theta]_0^\theta \Rightarrow \frac{v^2}{2} = R.g.(1-cos\theta)$$

So, the velocity of the mobile is given by the expression:

$$v = \sqrt{2.R.g(1-cos\theta)} \qquad (6)$$

Determining the relation of the reaction force *N* :

By replacing the equation (6) in (4), we get the expression of the reaction force N :

$$N = mg.(3cos\theta - 2)$$

c/ / Position of the mobile when leaving the trajectory:

The mobile leaves the circular trajectory when: $N = 0$.

According to the relation of the reaction force, we have:

$mg.(3cos\theta - 2) = 0 \Rightarrow (3cos\theta - 2) = 0 \Rightarrow cos\theta = \frac{2}{3}$ from where : $\theta = 48°$

Therefore, the mobile leaves the trajectory if the angle $\theta = 48°$

II/ Using the kinetic moment theorem :

a/ Study of the movement of the mobile:

By definition, the kinetic moment theorem is :

$$\frac{d\vec{L}_0}{dt} = \vec{M}_O(\sum \vec{F}_{ext}) = \sum \overrightarrow{OM} \wedge \vec{F}_{ext}$$

As well as, the kinetic moment relation, which is $\vec{L_0} = \overrightarrow{OM} \wedge \vec{P}$

So: $\vec{L_0} = \overrightarrow{OM} \wedge m\vec{v} = -R.\vec{u}_N \wedge mv.\vec{u}_T = R.m.v.\vec{u}_z$ (7)

If we derive the expression (7), we obtain:

$$\frac{d\vec{L_0}}{dt} = R.m.\frac{dv}{dt}.\vec{u}_z \qquad (8)$$

We have two forces exercised on the mobile, so we will have two moments of force: $\overrightarrow{OM} \wedge \vec{P} = -R.\vec{u}_N \wedge \left(mg.sin\theta.\vec{u}_T + mg.cos\theta.\vec{u}_N\right)$

$$\overrightarrow{OM} \wedge \vec{P} = R.m.g.sin\theta.\vec{u}_z \qquad (9)$$

$$\overrightarrow{OM} \wedge \vec{N} = -R.\vec{u}_N \wedge N.\vec{u}_N = \vec{0} \qquad (10)$$

b/ Velocity expression:

According to relations (7), (8) and (9), we will have:

$$\frac{dv}{dt} = g.sin\theta \Rightarrow \frac{dv}{d\theta}.\frac{d\theta}{dt} = g.sin\theta \ ;$$

Where: $\frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R}$

Hence: $\frac{v}{R}.\frac{dv}{d\theta} = g.sin\theta \Rightarrow v.dv = R.g.sin\theta.d\theta$ (11)

If we integrate the relation (11) :

$$\int_0^v v.dv = R.g.\int_0^\theta sin\theta.d\theta$$

$$\Rightarrow \frac{v^2}{2} = R.g.\left[-cos\theta\right]_0^\theta \Rightarrow \frac{v^2}{2} = R.g.(1-cos\theta)$$

Then the velocity of the mobile will be: $v = \sqrt{2.R.g(1-cos\theta)}$

Determining the relation of the reaction force *N*:

By replacing equation (6) in (4), we obtain: $N = mg(3cos\theta - 2)$

c/ Position of the mobile when leaving the trajectory:

The mobile leaves the surface when: $N = 0$; *so*: $\cos\theta = \frac{2}{3}$ *where*: $\theta = 48°$

d/ Calculation of the velocity when the mobile leaves the trajectory:

The result of the velocity is:

$$v = \sqrt{2.R.g(1 - \cos\theta)} = \sqrt{2.\ 1{,}36.\ 10(1 - \cos 48°)} = 2\ m/s$$