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1. Solutions of Tutorial Exercises (1):

A. LESLOUS **A. AYACHI**

Basic Definitions and Notation

Definition 1.1 (Set Complement):
Let E be a universal set and $A \subseteq E$. The **complement** of A in E, denoted $C_E A$ or $\overline{A}$, is defined as,

$$C_E A = \{x \in E \mid x \notin A\} = E \setminus A.$$

Definition 1.2 (Set Difference):
The **difference** of sets A and B, denoted $A \setminus B$, is defined as,

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

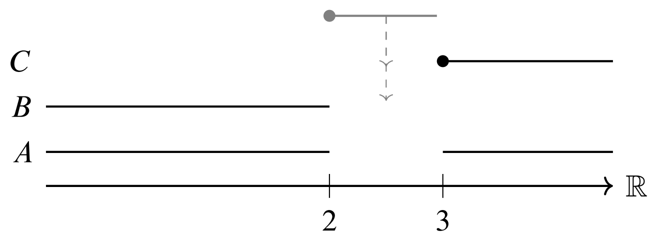
Set Complements and Properties

Exercise 1.1 (Sets and Complements):
The complement of A in E is denoted $C_E A$ (or $\bar{A}$), and is defined as the set of elements in E that are not elements of A.

1. Let $A =]-\infty, 2[\cup]3, \infty[$, $B =]-\infty, 2[$, and $C = [3, +\infty[$. Compare the following sets, $C_{\mathbb{R}} A$ and $C_{\mathbb{R}} B \cap C_{\mathbb{R}} C$.
2. Check that,
   (i) $A \subset B \Leftrightarrow C_E(B) \subset C_E(A)$
   (ii) $C_A(A \cap B) = A \setminus B$ where $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$
   To deduce that $A \setminus (A \setminus B) = A \cap B$
   (iii) $C_E(A \cup B) = C_E(A) \cap C_E(B)$
   (iv) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Solution:

1. **Comparison of $C_{\mathbb{R}} A$ and $C_{\mathbb{R}} B \cap C_{\mathbb{R}} C$**



- **Compute $C_{\mathbb{R}}A$,**

$$\begin{aligned} C_{\mathbb{R}}A &= \mathbb{R} \setminus A \\ &= \mathbb{R} \setminus (]-\infty, 2[\cup]3, \infty[) \\ &= [2, 3]. \end{aligned}$$

Justification: The union $]-\infty, 2[\cup]3, \infty[$ excludes exactly the closed interval $[2, 3]$.

- **Compute $C_{\mathbb{R}}B$,**

$$\begin{aligned} C_{\mathbb{R}}B &= \mathbb{R} \setminus B \\ &= \mathbb{R}\setminus]-\infty, 2[\\ &= [2, \infty[. \end{aligned}$$

- **Compute $C_{\mathbb{R}}C$,**

$$\begin{aligned} C_{\mathbb{R}}C &= \mathbb{R} \setminus C \\ &= \mathbb{R} \setminus [3, +\infty[\\ &=]-\infty, 3[. \end{aligned}$$

- **Compute the intersection,**

$$\begin{aligned} C_{\mathbb{R}}B \cap C_{\mathbb{R}}C &= [2, \infty[\cap]-\infty, 3[\\ &= [2, 3[, \end{aligned}$$

then,

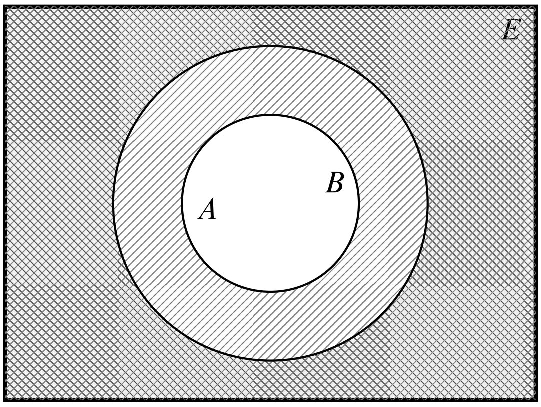
$$C_{\mathbb{R}}A = [2, 3] \quad \text{and} \quad C_{\mathbb{R}}B \cap C_{\mathbb{R}}C = [2, 3[.$$

Since $[2, 3[\subset [2, 3]$, we conclude,

$$\boxed{C_{\mathbb{R}}B \cap C_{\mathbb{R}}C \subset C_{\mathbb{R}}A.}$$

2. **Proof of Fundamental Properties**

   (i) **Property:** $A \subset B \Leftrightarrow C_E(B) \subset C_E(A)$.



Proof:
We prove both directions,

($\Rightarrow$) Assume $A \subset B$.

Let $x \in C_E(B)$. Then

$$x \in C_E(B) \iff x \in E \wedge x \notin B.$$

Since $A \subset B$, we have

$$x \notin B \implies x \notin A.$$

Therefore,

$$x \in E \wedge x \notin A \iff x \in C_E(A).$$

Hence $C_E(B) \subset C_E(A)$.

($\Leftarrow$) Assume $C_E(B) \subset C_E(A)$.

Let $x \in A$. Then

$$x \in A \iff x \in E \wedge x \notin C_E(A),$$

because $C_E(A) = E \setminus A$. □

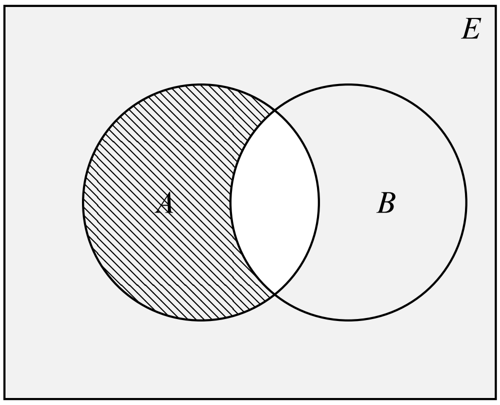
Also, since $C_E(B) \subset C_E(A)$, we have

$$x \notin C_E(A) \implies x \notin C_E(B) \iff x \in B.$$

Since this holds for all $x \in A$, we conclude that

$$A \subset B.$$

(ii) **Property:** $C_A(A \cap B) = A \setminus B$.



Proof:
By definition of relative complement,

$$C_A(A \cap B) = \{x \in A \mid x \notin (A \cap B)\}.$$

Since, for any x,

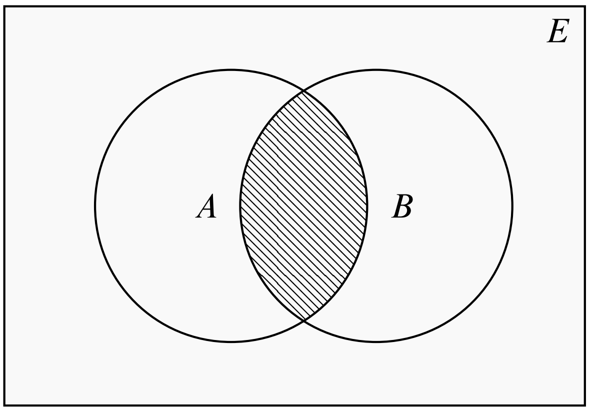
$$\begin{aligned} x \in C_A(A \cap B) &\iff x \in A \wedge x \notin (A \cap B) \\ &\iff x \in A \wedge (x \notin A \vee x \notin B) \\ &\iff (x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B) \\ &\iff \text{False} \vee x \in A \setminus B \\ &\iff x \in A \setminus B, \end{aligned}$$

we have,

$$\boxed{C_A(A \cap B) = \{x \in A \mid x \notin B\} = A \setminus B.}$$

□

Deduction: $A \setminus (A \setminus B) = A \cap B$.



Proof:
Using the result above, $A \setminus B = C_A(A \cap B)$. Then,

$$A \setminus (A \setminus B) = A \setminus C_A(A \cap B).$$

But for any x,

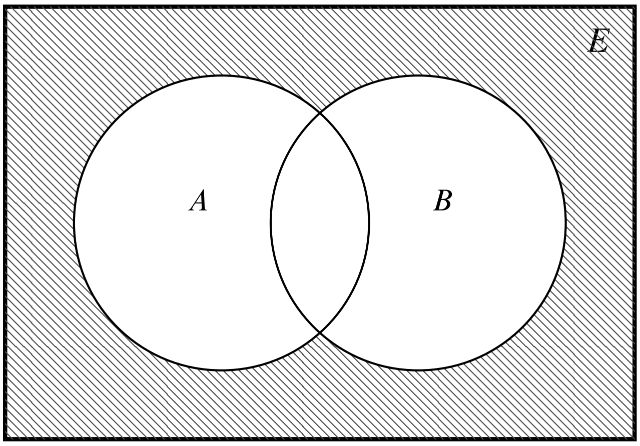
$$\begin{aligned} x \in A \setminus C_A(A \cap B) &\iff x \in A \wedge x \notin C_A(A \cap B) \\ &\iff x \in A \wedge x \in (A \cap B) \\ &\iff x \in A \cap A \cap B \\ &\iff x \in A \cap B. \end{aligned}$$

Thus,

$$\boxed{A \setminus (A \setminus B) = A \cap B.}$$

□

(iii) **Property:** $C_E(A \cup B) = C_E(A) \cap C_E(B)$ (De Morgan's Law),



Proof:
For any x , we have,

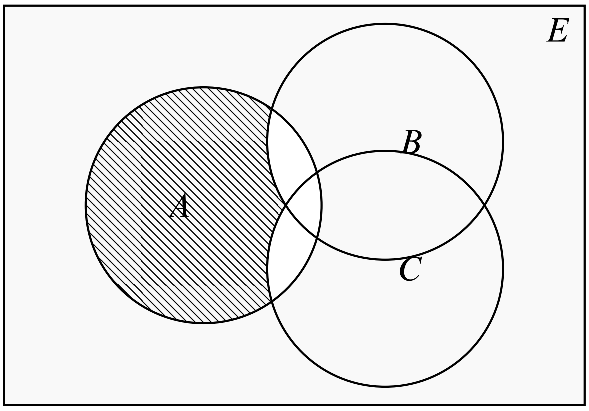
$$\begin{aligned} x \in C_E(A \cup B) &\iff x \in E \wedge x \notin (A \cup B) \\ &\iff x \in E \wedge (x \notin A \wedge x \notin B) \\ &\iff (x \in E \wedge x \notin A) \wedge (x \in E \wedge x \notin B) \\ &\iff x \in C_E(A) \wedge x \in C_E(B) \\ &\iff x \in C_E(A) \cap C_E(B), \end{aligned}$$

then

$$\boxed{C_E(A \cup B) = C_E(A) \cap C_E(B)}$$

□

(iv) **Property:** $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.



Proof:
Using Property (ii),

$$\begin{aligned} A \setminus (B \cup C) &= C_A(A \cap (B \cup C)) \\ &= C_A((A \cap B) \cup (A \cap C)) \\ &= C_A(A \cap B) \cap C_A(A \cap C) \text{ (Property (iii))} \\ &= (A \setminus B) \cap (A \setminus C) \text{ (Property (ii)).} \end{aligned}$$

Therefore,

$$\boxed{C_E(A \cup B) = C_E(A) \cap C_E(B)}$$

Alternatively: For any x , we have,

$$\begin{aligned} x \in A \setminus (B \cup C) &\iff x \in A \wedge x \notin (B \cup C) \\ &\iff x \in A \wedge (x \notin B \wedge x \notin C) \\ &\iff (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) \\ &\iff x \in (A \setminus B) \wedge x \in (A \setminus C) \\ &\iff x \in (A \setminus B) \cap (A \setminus C) \end{aligned}$$

□

Exercise 1.2:
Consider the sets,

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, \quad B = \{1, 4, 6, 7, 10, 14\}, \quad C = \{3, 5, 6, 7, 9\}, \quad \text{and} \quad D = \{0, 2, 4, 6, 8\}.$$

Find the following sets,

$A \cap C, B \cup D, B \cup D, A \cup B, C \cap D, B \cap C, A \cap (C \cup D), A \setminus D, A \setminus C, C \setminus B, (B \cup C) \setminus D$

Solution:

1. $A \cap C$ (elements in both A and C)

$$\begin{aligned} A \cap C \quad = \quad & \{0, 1, 2, \underbrace{3}, 4, \underbrace{5}, \underbrace{6}, \underbrace{7}, 8, \underbrace{9}\} \\ & \cap \{\underbrace{3}, \underbrace{5}, \underbrace{6}, \underbrace{7}, \underbrace{9}\} \end{aligned}$$

$$= \boxed{\{3, 5, 6, 7, 9\}.}$$

2. $B \cup D$ (all elements in B or D)

$$\begin{aligned} B \cup D &= \{1, \overbrace{4}, \overbrace{6}, 7, 10, 14\} \\ &\quad \cup \{0, 2, \overbrace{4}, \overbrace{6}, 8\} \\ &= \boxed{\{0, 1, 2, 4, 6, 7, 8, 10, 14\}.} \end{aligned}$$

3. $A \cup B$ (all elements in A or B)

$$\begin{aligned} A \cup B &= \{0, \overbrace{1}, 2, 3, \overbrace{4}, 5, \overbrace{6}, \overbrace{7}, 8, 9\} \\ &\quad \cup \{\overbrace{1}, \overbrace{4}, \overbrace{6}, \overbrace{7}, 10, 14\} \\ &= \boxed{\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 14\}.} \end{aligned}$$

4. $C \cap D$ (elements in both C and D)

$$\begin{aligned} C \cap D &= \{3, 5, \underbrace{6}, 7, 9\} \cap \{0, 2, 4, \underbrace{6}, 8\} \\ &= \boxed{\{6\}.} \end{aligned}$$

5. $(B \cap C) \cap D$ (elements in B and C, then also in D)

$$\begin{aligned} (B \cap C) \cap D &= B \cap (C \cap D) \\ &\quad = \{1, 4, \underbrace{6}, 7, 10, 14\} \cap \{\underbrace{6}\} \\ &\quad = \boxed{\{6\}.} \end{aligned}$$

Equivalently,

$$\begin{aligned} B \cap C &= \{1, 4, \underbrace{6}, \underbrace{7}, 10, 14\} \cap \{3, 5, \underbrace{6}, \underbrace{7}, 9\} = \{6, 7\} \\ &\implies (B \cap C) \cap D = \{\underbrace{6}, 7\} \cap \{0, 2, 4, \underbrace{6}, 8\} \\ &= \boxed{\{6\}.} \end{aligned}$$

6. $A \cap (C \cup D)$ (elements in A and also in C or D)

$$\begin{aligned} C \cup D &= \{3, 5, 6, 7, 9\} \cup \{0, 2, 4, 6, 8\} \\ &= \{0, 2, 3, 4, 5, 6, 7, 8, 9\} \\ \implies \quad & A \cap (C \cup D) = \{\overbrace{0}, 1, \overbrace{2, 3, 4, 5, 6, 7, 8, 9}\} \cap \{\overbrace{0}, \overbrace{2, 3, 4, 5, 6, 7, 8, 9}\} \\ &= \boxed{\{0, 2, 3, 4, 5, 6, 7, 8, 9\}.} \end{aligned}$$

7. $A \setminus D$ (elements in A but not in D)

$$\begin{aligned} A \setminus D &= \{\underbrace{0}, 1, \underbrace{2}, 3, \underbrace{4}, 5, \underbrace{6}, 7, \underbrace{8}, 9\} \setminus \{0, 2, 4, 6, 8\} \\ &= \boxed{\{1, 3, 5, 7, 9\}.} \end{aligned}$$

8. $A \setminus C$ (elements in A but not in C)

$$\begin{aligned} A \setminus C &= \{0, 1, 2, \underbrace{3}, 4, \underbrace{5}, \underbrace{6}, \underbrace{7}, 8, \underbrace{9}\} \setminus \{3, 5, 6, 7, 9\} \\ &= \{0, 1, 2, 4, 8\} \end{aligned}$$

9. $C \setminus B$ (elements in C but not in B)

$$\begin{aligned} C \setminus B &= \{3, 5, \underbrace{6}, 7, 9\} \setminus \{1, 4, \underbrace{6}, 7, 10, 14\} \\ &= \boxed{\{3, 5, 9\}.} \end{aligned}$$

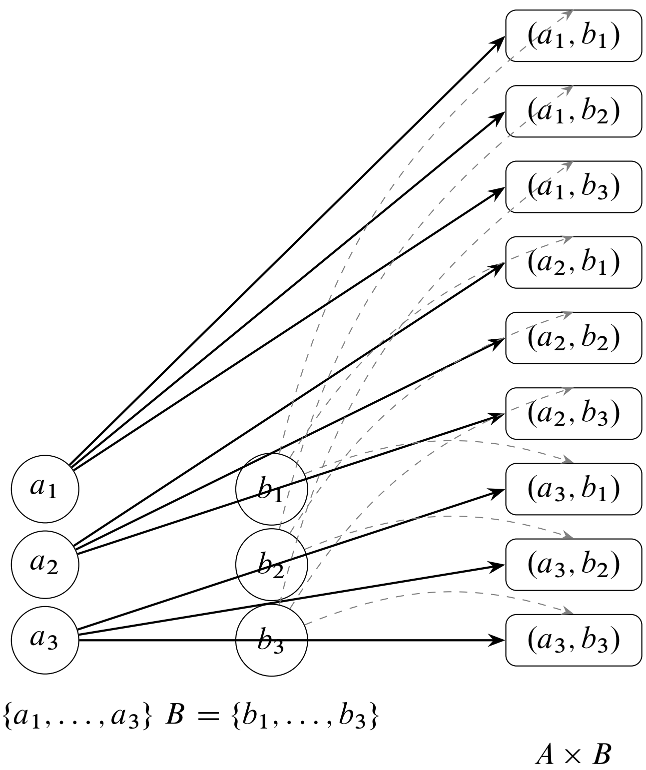
10. $(B \cup C) \setminus D$ (all elements in B or C, but not in D)

$$\begin{aligned} B \cup C \setminus D &= \{1, 4, \underbrace{6}, \underbrace{7}, 10, 14\} \cup \{3, 5, \underbrace{6}, \underbrace{7}, 9\} \setminus D \\ &= \{1, 3, \underbrace{4}, 5, \underbrace{6}, 7, 9, 10, 14\} \setminus \{0, 2, 4, 6, 8\} \\ &= \boxed{\{1, 3, 5, 7, 9, 10, 14\}.} \end{aligned}$$

Definition 1.3 (Cartesian product):
Given sets $A_1, A_2, \dots, A_n$, their **Cartesian product** is defined as,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \ \text{ for each } i = 1, \dots, n\}.$$

- For $n = 2$: Ordered pairs (a, b) where $a \in A, b \in B$
- For $n = 3$: Ordered triples (a, b, c) where $a \in A, b \in B, c \in C$



- For general n: Ordered n-tuples $(a_1, \dots, a_n)$

Example 1.1:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$
$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$$
$$[a, b] \times [c, d] = \{(x, y) \mid x \in [a, b], y \in [c, d]\}$$

Exercise 1.3:
Let

$$A = \{1, 3, 5\}, \quad B = \{a, b\}.$$

1. Find $A \times B$ and $B \times A$.
2. Are $A \times B$ and $B \times A$ equal? Justify your answer.

Solution:

1. **Finding** $A \times B$ **and** $B \times A$.

   Proof:
   Recall the definition (law of Cartesian product),

   $$A \times B = \{(a, b) \mid a \in A, b \in B\}, \quad B \times A = \{(b, a) \mid b \in B, a \in A\}$$

   Apply the law to $A \times B$,

   $$x = 1 \in A \implies (1, a), (1, b)$$
   $$x = 3 \in A \implies (3, a), (3, b)$$
   $$x = 5 \in A \implies (5, a), (5, b)$$
   $$\implies \boxed{A \times B = \{(1, a), (1, b), (3, a), (3, b), (5, a), (5, b)\}.}$$

   Apply the law to $B \times A$

   $$x = a \in B \implies (a, 1), (a, 3), (a, 5)$$
   $$x = b \in B \implies (b, 1), (b, 3), (b, 5)$$
   $$\implies \boxed{B \times A = \{(a, 1), (a, 3), (a, 5), (b, 1), (b, 3), (b, 5)\}.}$$

   □

2. **Comparing** $A \times B$ **and** $B \times A$.

   By definition of equality of sets, two sets are equal if every element of one set is in the other.

   Consider $(1, a) \in A \times B$. Clearly, $(1, a) \notin B \times A$ because in $B \times A$ the first component must come from B.

   Therefore,

   $$\boxed{A \times B \neq B \times A.}$$

   So, Cartesian product is not commutative.

Exercise 1.4:
The symmetric difference between A and B, denoted $A \triangle B$, is the set of elements that are in one and only one of the sets.

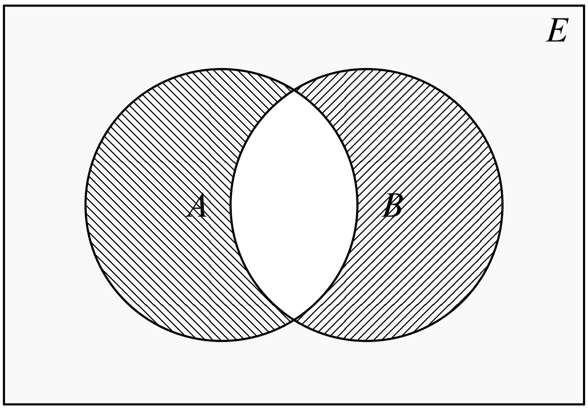
By definition, we have

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = \{ x \in E \mid (x \in A \wedge x \notin B) \ \vee \ (x \in B \wedge x \notin A) \}.$$

Check that

$$A \triangle B = (A \cap C_E(B)) \cup (B \cap C_E(A)) = (A \cup B) \setminus (A \cap B).$$

Solution:



Proof:
Let $A, B \subseteq E$. For any x we have,

$$\begin{aligned}
x \in (A \cup B) \setminus (A \cap B) &\iff x \in (A \cup B) \wedge x \notin (A \cap B) \\
&\iff (x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B) \\
&\iff (x \in A \wedge (x \notin A \vee x \notin B)) \vee (x \in B \wedge (x \notin A \vee x \notin B)) \\
&\iff (x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B)) \\
&\quad \vee ((x \in B \wedge x \notin B) \vee (x \in B \wedge x \notin A)) \\
&\iff (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \\
&\iff x \in (A \setminus B) \cup (B \setminus A) \qquad \text{(definition of set difference } \triangle) \\
&\iff x \in (A \setminus B) \ \vee \ x \in (B \setminus A) \\
&\iff (x \in A \wedge x \notin B) \ \vee \ (x \in B \wedge x \notin A) \\
&\iff x \in (A \cap C_E(B)) \cup (B \cap C_E(A)) \qquad \text{(using } C_E(\cdot))
\end{aligned}$$

Since the membership condition for an arbitrary x is the same on both sides, we conclude

$$\boxed{(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A) = (A \cap C_E(B)) \cup (B \cap C_E(A)).}$$

□

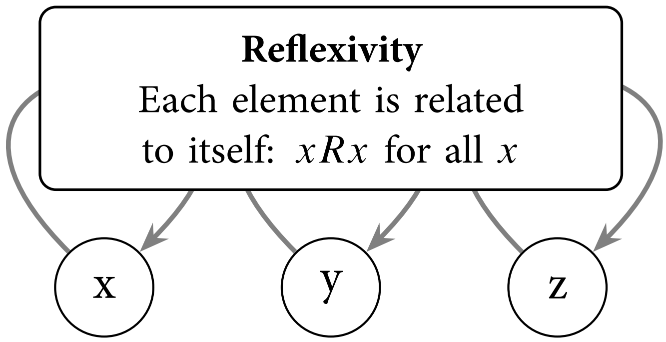
2. Solutions of Tutorial Exercises (2):

A. LESLOUS A. AYACHI

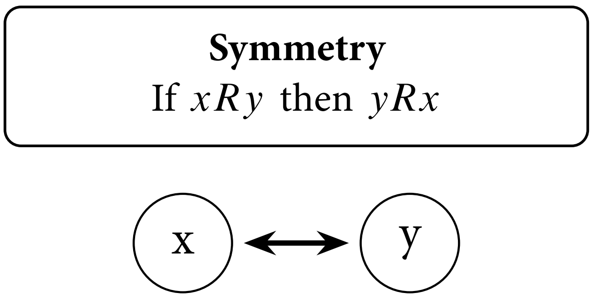
Theoretical Background

Definition 2.1 (Equivalence Relation):
A relation R on a set E is an **equivalence relation** if it satisfies the following properties,

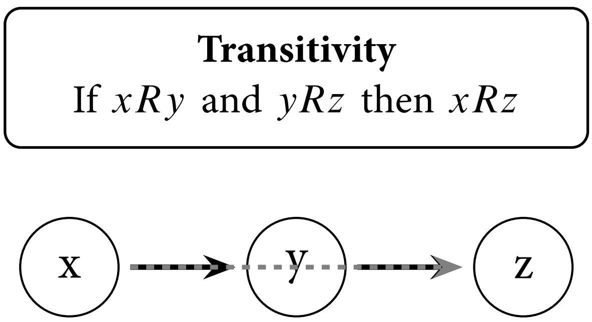
(a) **Reflexivity**: $\forall x \in E,\ xRx$



(b) **Symmetry**: $\forall x, y \in E,\ xRy \Rightarrow yRx$



(c) **Transitivity**: $\forall x, y, z \in E,\ (xRy \wedge yRz) \Rightarrow xRz$

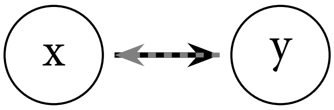


Definition 2.2 (Order Relation):
A relation R on a set E is an **order relation** if it satisfies the following properties,

(i) **Reflexivity**: $\forall x \in E,\ xRx$

(ii) **Antisymmetry**: $\forall x, y \in E,\ (xRy \wedge yRx) \Rightarrow x = y$

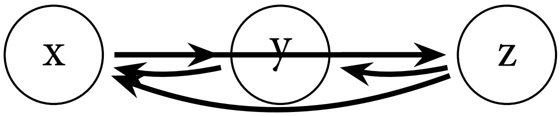
Antisymmetry
If xRy and yRx then $x = y$



(iii) **Transitivity**: $\forall x, y, z \in E,\ (xRy \wedge yRz) \Rightarrow xRz$

If, in addition,

(iv) **Total Comparability:** $\forall x, y \in E,\ xRy \vee yRx$, then the order is **total**.



Total Comparability
For every pair x, y, either
xRy or yRx holds
(All elements are comparable)

Exercise 2.1:
Let R be a relation on $\mathbb{Z}$ defined by,

$$\forall x, y \in \mathbb{Z}, \quad xRy \iff x^2 - y^2 = x - y.$$

1. Show that R is an equivalence relation.
2. Determine the equivalence classes for all elements.

Solution:

1. **Equivalence Relation:**
   (a) **Reflexivity**: $\forall x \in \mathbb{Z}$, does xRx hold?

$$x^2 - x^2 = 0 = x - x \Rightarrow xRx.$$

   (b) **Symmetry**: $\forall x, y \in \mathbb{Z}$, if xRy, does yRx hold?

$$\begin{aligned} xRy &\iff x^2 - y^2 = x - y \\ &\iff -(y^2 - x^2) = -(y - x) \end{aligned}$$

$$\begin{aligned} &\iff y^2 - x^2 = y - x \\ &\iff yRx. \end{aligned}$$

(c) **Transitivity**: $\forall x, y, z \in \mathbb{Z}$, if xRy and yRz, does xRz hold?

$$\begin{aligned} xRy &\iff x^2 - y^2 = x - y \quad (1) \\ yRz &\iff y^2 - z^2 = y - z \quad (2) \end{aligned}$$

Adding (1) and (2),

$$\begin{aligned} (x^2 - y^2) + (y^2 - z^2) &= (x - y) + (y - z) \\ x^2 - z^2 &= x - z \\ &\Rightarrow xRz. \end{aligned}$$

Thus,

$$\boxed{R \text{ is an equivalence relation.}}$$

2. **Equivalence Classes:**

For $a \in \mathbb{Z}$, the equivalence class is,

$$\begin{aligned} [a] &= \{x \in \mathbb{Z} \mid xRa\} \\ &= \{x \in \mathbb{Z} \mid x^2 - a^2 = x - a\} \\ &= \{x \in \mathbb{Z} \mid (x - a)(x + a) = x - a\} \\ &= \{x \in \mathbb{Z} \mid (x - a)(x + a) - (x - a) = 0\} \\ &= \{x \in \mathbb{Z} \mid (x - a)(x + a - 1) = 0\} \\ &= \{x \in \mathbb{Z} \mid x = a \vee x = 1 - a\} \\ &= \boxed{\{a, 1 - a\}.} \end{aligned}$$

Exercise 2.2:
Let R be a relation on $\mathbb{R}$ defined by,

$$\forall x, y \in \mathbb{R}, \quad xRy \iff \cos^2(x) + \sin^2(y) = 1.$$

Show that R is an equivalence relation.

Solution:

(a) **Reflexivity**: $\forall x \in \mathbb{R}$, does xRx hold?

$$\cos^2(x) + \sin^2(x) = 1 \Rightarrow xRx.$$

(b) **Symmetry**: $\forall x, y \in \mathbb{R}$, if xRy, does yRx hold?

$$\begin{aligned} xRy &\iff \cos^2(x) + \sin^2(y) = 1 \\ &\iff 1 - \sin^2(x) + 1 - \cos^2(y) = 2 - (\sin^2(x) + \cos^2(y)) \\ &\iff \cos^2(y) + \sin^2(x) = 1 \\ &\iff yRx. \end{aligned}$$

(c) **Transitivity**: $\forall x, y, z \in \mathbb{R}$, if xRy and yRz, does xRz hold?

We have,

$$xRy \iff \cos^2(x) + \sin^2(y) = 1 \quad (1)$$
$$yRz \iff \cos^2(y) + \sin^2(z) = 1 \quad (2)$$

From (1): $\sin^2(y) = 1 - \cos^2(x)$
From (2): $\cos^2(y) = 1 - \sin^2(z)$
But also from (1): $\cos^2(x) = 1 - \sin^2(y)$
Substituting,

$$\begin{aligned}\cos^2(x) + \sin^2(z) &= (1 - \sin^2(y)) + \sin^2(z)\\ &= 1 - \sin^2(y) + \sin^2(z)\\ &= 1 - (1 - \cos^2(y)) + \sin^2(z)\\ &= \cos^2(y) + \sin^2(z)\\ &= 1 \quad \text{(from (2))}\\ &\Rightarrow xRz.\end{aligned}$$

Thus,

$$\boxed{R \text{ is an equivalence relation.}}$$

Exercise 2.3:
Let R be a relation on $\mathbb{N}^*$ defined by,

$$aRb \iff \exists k \in \mathbb{N}^*,\ b = a^k.$$

1. Show that R is an order relation.
2. Is it a total order?

Solution:

1. **Order Relation:**

(i) **Reflexivity**: $\forall a \in \mathbb{N}^*$, does aRa hold?

$$a = a^1 \Rightarrow \boxed{aRa.}$$

(ii) **Antisymmetry**: $\forall a, b \in \mathbb{N}^*$, if aRb and bRa, does $a = b$ hold?

$$aRb \Rightarrow \exists k_1 \in \mathbb{N}^*,\ b = a^{k_1} \quad (1)$$
$$bRa \Rightarrow \exists k_2 \in \mathbb{N}^*,\ a = b^{k_2} \quad (2)$$

Substituting (1) into (2),

$$\begin{aligned}a &= (a^{k_1})^{k_2}\\ a &= a^{k_1 k_2}\\ &\Rightarrow k_1 k_2 = 1\end{aligned}$$

$$\Rightarrow k_1 = \frac{1}{k_2} \in \mathbb{N}^*$$
$$\Rightarrow k_1 = 1 \wedge k_2 = 1$$
$$\Rightarrow \boxed{b = a^1 = a.}$$

(iii) **Transitivity**: $\forall a, b, c \in \mathbb{N}^*$, if aRb and bRc, does aRc hold?

$$aRb \Rightarrow \exists k_1 \in \mathbb{N}^*,\ b = a^{k_1} \quad (1)$$
$$bRc \Rightarrow \exists k_2 \in \mathbb{N}^*,\ c = b^{k_2} \quad (2)$$

Substituting (1) into (2),

$$c = (a^{k_1})^{k_2}$$
$$c = a^{k_1 k_2}$$
$$\Rightarrow \exists k_3 = k_1 k_2 \in \mathbb{N}^*,\ c = a^{k_3}$$
$$\iff aRc$$

Thus,

$$\boxed{R \text{ is an order relation.}}$$

2. **Total Order:**

A total order requires that every pair of distinct elements is comparable, i.e.,

$$\forall a, b \in \mathbb{N}^*,\ aRb \vee bRa.$$

To disprove totality, we must show,

$$\exists a, b \in \mathbb{N}^*,\ (a \not R b) \wedge (b \not R a).$$

Counterexample: $a = 2, b = 3$

$$(2 \not R 3) \iff \forall k \in \mathbb{N}^*,\ 3 \neq 2^k,\ \text{which is true}$$
$$(3 \not R 2) \iff \forall k \in \mathbb{N}^*,\ 2 \neq 3^k,\ \text{which is true}$$
$$\Rightarrow \boxed{R \text{ is not total.}}$$

The order is **partial**.

Exercise 2.4:

Let R be a relation on $\mathbb{R}^2$ defined by,

$$(x_1, y_1) R (x_2, y_2) \iff x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

1. Show that R is an order relation.
2. Is it a total order?

Solution:

1. **Order Relation:**

(i) **Reflexivity**: $\forall(x, y) \in \mathbb{R}^2$, does $(x, y)R(x, y)$ hold?

$$x \le x \ \wedge \ y \le y \Rightarrow \boxed{(x, y)R(x, y).}$$

(ii) **Antisymmetry**: $\forall(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, if $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_1, y_1)$, does $(x_1, y_1) = (x_2, y_2)$ hold?

We have,

$$(x_1, y_1)R(x_2, y_2) \Rightarrow x_1 \le x_2 \ \wedge \ y_1 \le y_2 \quad (1)$$
$$(x_2, y_2)R(x_1, y_1) \Rightarrow x_2 \le x_1 \ \wedge \ y_2 \le y_1 \quad (2)$$

From (1) and (2),

$$x_1 \le x_2 \ \wedge \ x_2 \le x_1 \Rightarrow x_1 = x_2$$
$$y_1 \le y_2 \ \wedge \ y_2 \le y_1 \Rightarrow y_1 = y_2$$
$$\Rightarrow \boxed{(x_1, y_1) = (x_2, y_2).}$$

(iii) **Transitivity**: $\forall(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$, if $(x_1, y_1)R(x_2, y_2)$ and $(x_2, y_2)R(x_3, y_3)$, does $(x_1, y_1)R(x_3, y_3)$ hold?

We have,

$$(x_1, y_1)R(x_2, y_2) \Rightarrow x_1 \le x_2 \ \wedge \ y_1 \le y_2 \quad (1)$$
$$(x_2, y_2)R(x_3, y_3) \Rightarrow x_2 \le x_3 \ \wedge \ y_2 \le y_3 \quad (2)$$

From (1) and (2),

$$x_1 \le x_2 \ \wedge \ x_2 \le x_3 \Rightarrow x_1 \le x_3$$
$$y_1 \le y_2 \ \wedge \ y_2 \le y_3 \Rightarrow y_1 \le y_3$$
$$\Rightarrow \boxed{(x_1, y_1)R(x_3, y_3).}$$

Thus, R is an order relation.

2. **Total Order:**

A total order requires that every pair of elements is comparable, i.e.,

$$\forall(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2, \ (x_1, y_1)R(x_2, y_2) \vee (x_2, y_2)R(x_1, y_1).$$

To disprove totality, we must show,

$$\exists(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2, \ (x_1, y_1)\not{R}(x_2, y_2) \wedge (x_2, y_2)\not{R}(x_1, y_1).$$

Counterexample: $(1, 2)$ and $(2, 1)$

$$(1, 2)\not{R}(2, 1) \iff 1 \not\le 2 \ \vee \ 2 \not\le 1 \iff \text{True} \ \vee \ \text{False} \iff \text{True}$$
$$(2, 1)\not{R}(1, 2) \iff 2 \not\le 1 \ \vee \ 1 \not\le 2 \iff \text{True} \ \vee \ \text{False} \iff \text{True.}$$

$$\boxed{\text{The order is partial}}$$

3. Solutions of Tutorial Exercises (3):

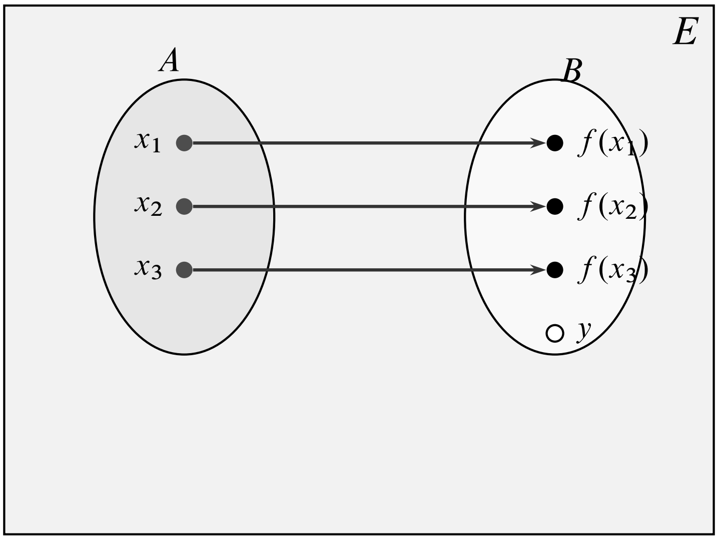
A. LESLOUS **A. AYACHI**

Mathematical Definitions

Definition 3.1 (Injective Function (One-to-One)):
A function $f : A \to B$ is **injective** if,

$$\forall x_1, x_2 \in A, \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2,$$

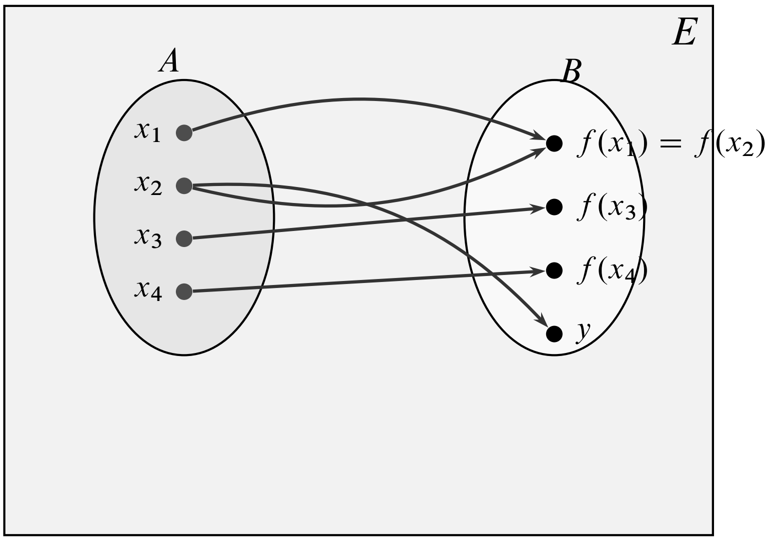
i.e., each $y \in B$ is mapped to by at most one $x \in A$.



Definition 3.2 (Surjective Function (Onto)):
A function $f : A \to B$ is **surjective** if,

$$\forall y \in B, \exists x \in A \text{ such that } f(x) = y,$$

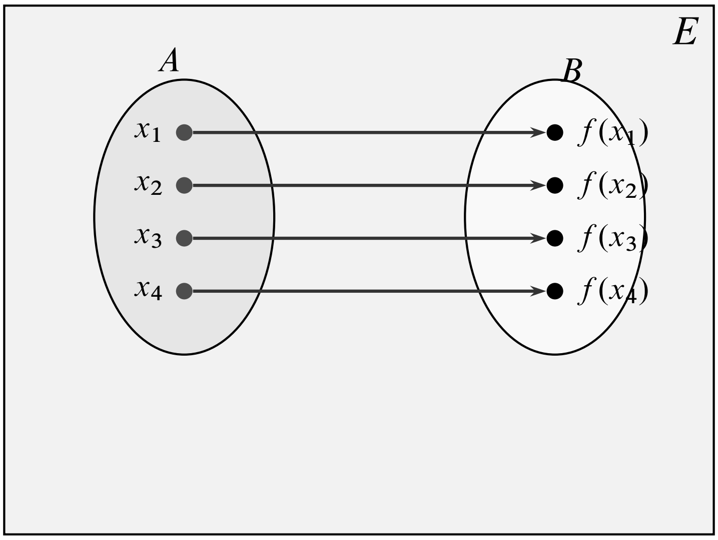
i.e., each $y \in B$ is mapped to by at least one $x \in A$.



Definition 3.3 (Bijective Function):
A function is **bijective** if it is both injective and surjective,

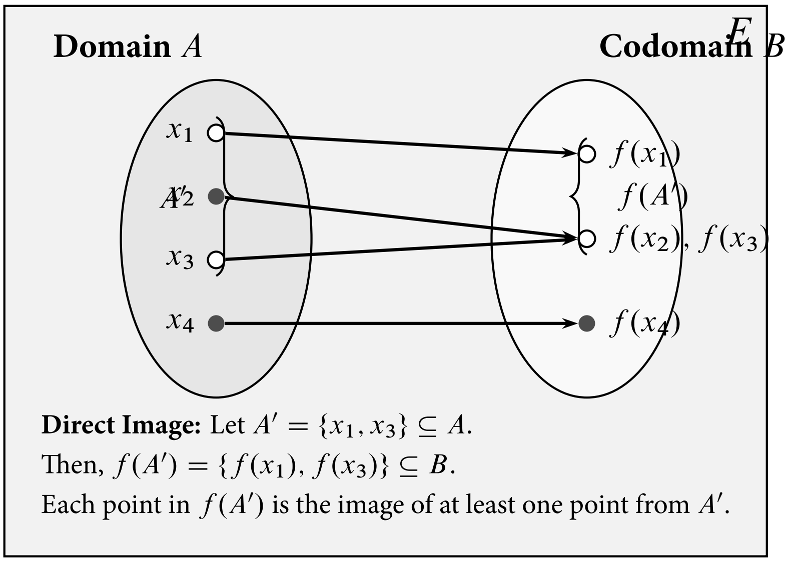
$$\forall y \in B, \text{ there exists exactly one} x \in A \text{ such that } f(x) = y,$$

*i.e., every element in the codomain is mapped to by at **least one** and at **most one** element in the domain.*



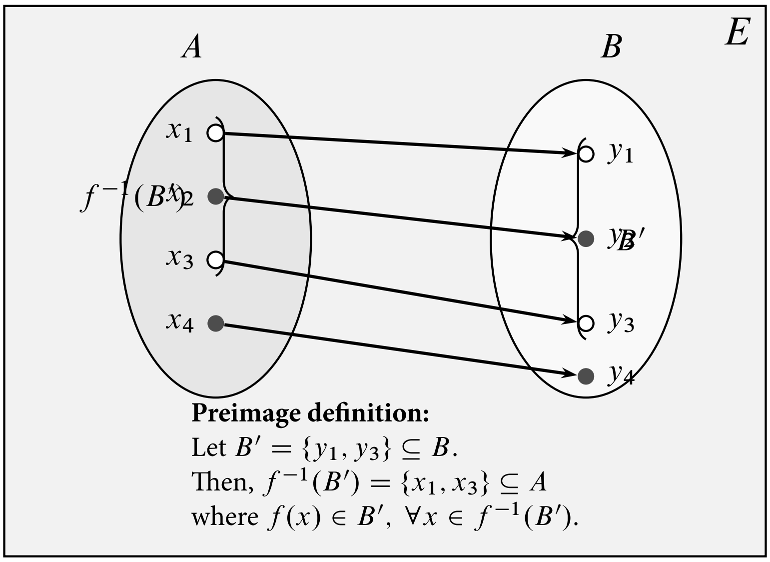
Definition 3.4 (Direct Image):
For $A' \subseteq A$, the **direct image** is,

$$f(A') = \{f(x) \in B \mid x \in A'\}.$$



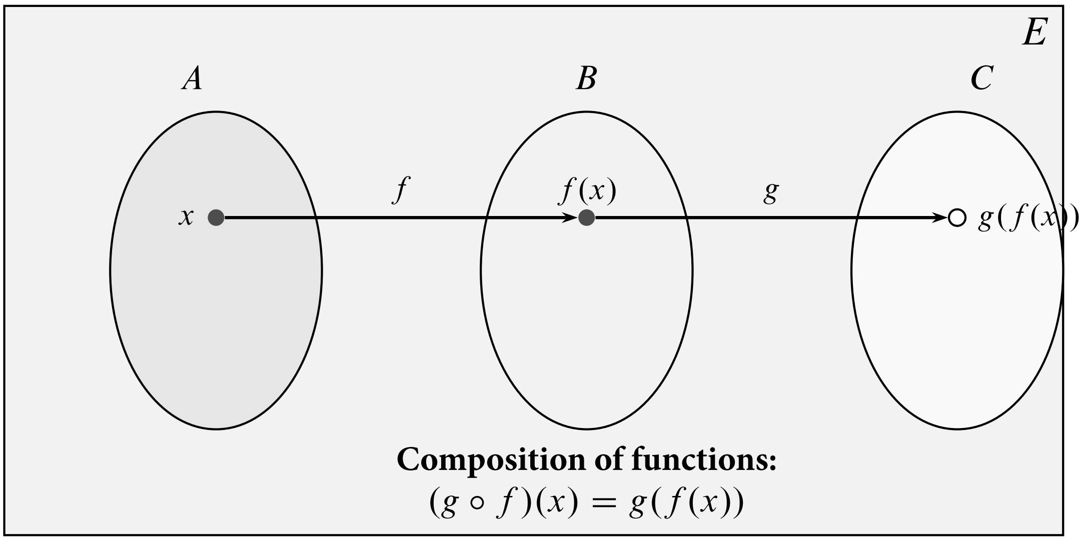
Definition 3.5 (Preimage (Inverse Image)):
For $B' \subseteq B$, the **preimage** is,

$$f^{-1}(B') = \{x \in A \mid f(x) \in B'\}.$$



Definition 3.6 (Function Composition):
For $f : A \to B$ and $g : B \to C$, the **composition** is,

$$(g \circ f)(x) = g(f(x)).$$



***Remark* 3.1 (Inverse Function):**
If $f : A \to B$ is bijective, its **inverse** $f^{-1} : B \to A$ satisfies,

$f^{-1}(f(x)) = x$ for all $x \in A$

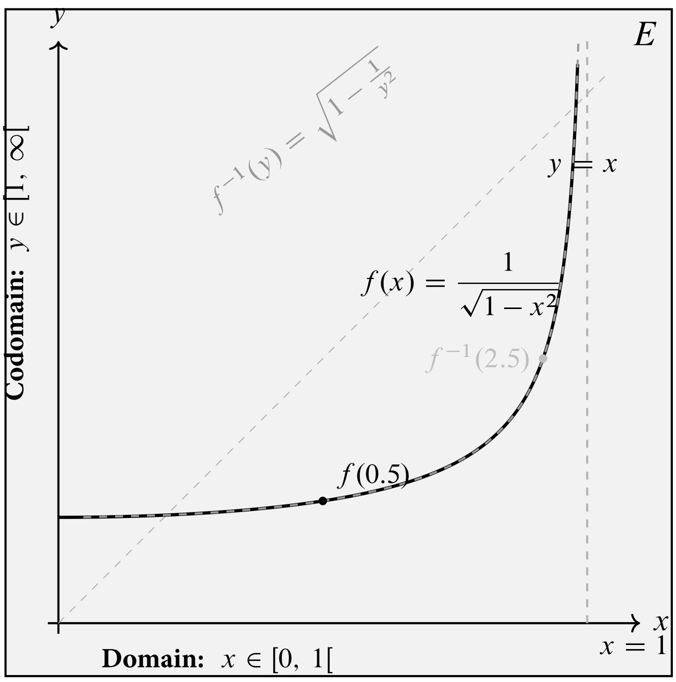
$f(f^{-1}(y)) = y$ for all $y \in B$

Exercise 3.1:
Let $f : [0, 1[\to [1, +\infty[$ be defined by

$$f(x) = \frac{1}{\sqrt{1 - x^2}}.$$

Is f bijective? give the inverse of f.

Solution:



- **Bijectivity**

  1. **Injectivity**

     Let $x_1, x_2 \in [0, 1[$ such that $f(x_1) = f(x_2)$. Then,

$$
\begin{aligned}
f(x_1) = f(x_2) &\iff \frac{1}{\sqrt{1 - x_1^2}} = \frac{1}{\sqrt{1 - x_2^2}} \\
&\iff \sqrt{1 - x_1^2} = \sqrt{1 - x_2^2} \quad \text{(since denominators are positive)} \\
&\iff 1 - x_1^2 = 1 - x_2^2 \quad \text{(squaring both sides)} \\
&\iff x_1^2 = x_2^2 \\
&\iff |x_1| = |x_2|
\end{aligned}
$$

     Since $x_1, x_2 \in [0, 1[$, we have $|x_1| = x_1$ and $|x_2| = x_2$, therefore $x_1 = x_2$.

     Thus,

$$\boxed{f \text{ is injective.}}$$

  2. **Surjectivity**

     Let $y \in [1, +\infty[$. We seek $x \in [0, 1[$ such that $f(x) = y$,

$$
\begin{aligned}
f(x) = y &\iff \frac{1}{\sqrt{1 - x^2}} = y \\
&\iff \sqrt{1 - x^2} = \frac{1}{y} \quad \text{(since } y \geq 1 > 0\text{)} \\
&\iff 1 - x^2 = \frac{1}{y^2} \quad \text{(squaring both sides)} \\
&\iff x^2 = 1 - \frac{1}{y^2} \\
&\iff x = \sqrt{1 - \frac{1}{y^2}} \quad \text{(taking positive root since } x \geq 0\text{)}
\end{aligned}
$$

     Verification,

     - **Non-negativity**: Since $y \geq 1$, we have $1 - \frac{1}{y^2} \geq 0$, so $x \geq 0$.
     - **Upper bound**: For $y \geq 1$, we have $\frac{1}{y^2} > 0$, so $x^2 = 1 - \frac{1}{y^2} < 1$, hence $x < 1$.

     Therefore, for every $y \in [1, +\infty[$, there exists a unique $x \in [0, 1[$ given by $x = \sqrt{1 - \frac{1}{y^2}}$.

     Thus,

$$\boxed{f \text{ is surjective.}}$$

- **Inverse Function**

From the surjectivity proof, we obtain the inverse mapping,

$$\boxed{f^{-1}(y) = \sqrt{1 - \frac{1}{y^2}}, \quad y \in [1, +\infty[.}$$

- **Verification of Inverse Relationship**

  We verify that $f^{-1} \circ f = \mathrm{id}_{[0,1[}$ and $f \circ f^{-1} = \mathrm{id}_{[1,+\infty[}$,

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \sqrt{1 - \frac{1}{\left(\frac{1}{\sqrt{1-x^2}}\right)^2}} \\ &= \sqrt{1-(1-x^2)} = \sqrt{x^2} = x \quad \text{for } x \in [0, 1[,\end{aligned}$$

$$\begin{aligned}(f \circ f^{-1})(y) &= f\left(\sqrt{1 - \frac{1}{y^2}}\right) = \frac{1}{\sqrt{1 - \left(1 - \frac{1}{y^2}\right)}} \\ &= \frac{1}{\sqrt{\frac{1}{y^2}}} = y \quad \text{for } y \in [1, +\infty[.\end{aligned}$$

Exercise 3.2:

Consider $h : \mathbb{R} \to \mathbb{R}$ given by

$$h(x) = \frac{4x}{x^2+1}.$$

(1) Verify that for all $a \in \mathbb{R}^*$, we have $h(a) = h\left(\frac{1}{a}\right)$.

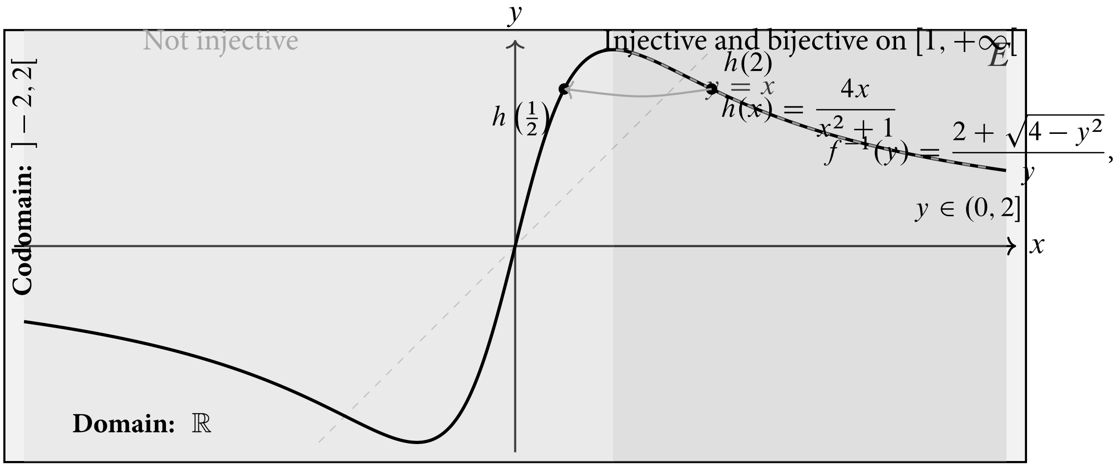
(2) Is h injective? Justify your answer.

Now define f on the interval $I = [1, +\infty[$ by $f(x) = h(x)$.

(3) Prove that f is injective.

(4) Show that f is bijective from $I \to]0, 2]$, and determine $f^{-1}(x)$.

Solution:



1. For $a \in \mathbb{R}^*$, we compute,

$$h\left(\frac{1}{a}\right) = \frac{4 \cdot \frac{1}{a}}{\left(\frac{1}{a}\right)^2 + 1} = \frac{\frac{4}{a}}{\frac{1}{a^2} + 1} = \frac{\frac{4}{a}}{\frac{1+a^2}{a^2}} = \frac{4}{a} \cdot \frac{a^2}{1 + a^2} = \frac{4a}{1 + a^2} = h(a)$$

2. **Injectivity of h**

   The symmetry property $h(a) = h(1/a)$ immediately implies non-injectivity. For example,

$$h(2) = \frac{4 \cdot 2}{2^2 + 1} = \frac{8}{5}$$
$$h\left(\frac{1}{2}\right) = \frac{4 \cdot \frac{1}{2}}{\left(\frac{1}{2}\right)^2 + 1} = \frac{2}{\frac{1}{4} + 1} = \frac{2}{\frac{5}{4}} = \frac{8}{5}$$

   Since $2 \neq \frac{1}{2}$ but $h(2) = h\left(\frac{1}{2}\right)$,

$$\boxed{h \text{is not injective on } \mathbb{R}.}$$

Part 2. Define $f : [1, +\infty[\to \mathbb{R}$ by $f(x) = h(x)$.

3. **Injectivity of f on** $[1, +\infty[$

   Let $x_1, x_2 \in [1, +\infty[$ with $f(x_1) = f(x_2)$,

$$\begin{aligned}
f(x_1) = f(x_2) &\iff \frac{4x_1}{x_1^2 + 1} = \frac{4x_2}{x_2^2 + 1} \\
&\iff 4x_1(x_2^2 + 1) = 4x_2(x_1^2 + 1) \\
&\iff x_1(x_2^2 + 1) = x_2(x_1^2 + 1) \\
&\iff x_1x_2^2 + x_1 = x_1^2x_2 + x_2 \\
&\iff x_1x_2^2 - x_1^2x_2 + x_1 - x_2 = 0 \\
&\iff x_1x_2(x_2 - x_1) - (x_2 - x_1) = 0 \\
&\iff (x_2 - x_1)(x_1x_2 - 1) = 0
\end{aligned}$$

   This gives $x_2 - x_1 = 0$ or $x_1x_2 = 1$. For $x_1, x_2 \geq 1$, $x_1x_2 = 1$ implies $x_1 = x_2 = 1$. Thus, in both cases, $x_1 = x_2$, proving

$$\boxed{f \text{ is injective on } [1, +\infty[.}$$

4. **Bijectivity and Inverse Function**

Let $y \in (0, 2]$. We solve $f(x) = y$ for $x \geq 1$,

$$f(x) = y \iff \frac{4x}{x^2+1} = y$$
$$\iff yx^2 + y = 4x$$
$$\iff yx^2 - 4x + y = 0.$$

The discriminant is,

$$\Delta = (-4)^2 - 4 \cdot y \cdot y = 16 - 4y^2 = 4(4 - y^2).$$

Since $y \in (0, 2]$, $\Delta \geq 0$. The roots are,

$$x = \frac{4 \pm \sqrt{4(4-y^2)}}{2y} = \frac{4 \pm 2\sqrt{4-y^2}}{2y} = \frac{2 \pm \sqrt{4-y^2}}{y}.$$

Let $x_1 = \frac{2+\sqrt{4-y^2}}{y}$, $x_2 = \frac{2-\sqrt{4-y^2}}{y}$.

- For x_1: Since $y \leq 2$, $\frac{2}{y} \geq 1$, and $\sqrt{4-y^2} \geq 0$, so $x_1 \geq 1$.
- For x_2: Since $\sqrt{4-y^2} \leq 2$, $x_2 \leq 1$ (equality only when $y = 2$).

Thus, we select x_1 as the valid solution. The inverse function is,

$$\boxed{f^{-1}(y) = \frac{2+\sqrt{4-y^2}}{y}, \quad y \in (0, 2].}$$

Verification,

$$(f \circ f^{-1})(y) = f\left(\frac{2+\sqrt{4-y^2}}{y}\right) = \frac{4 \cdot \frac{2+\sqrt{4-y^2}}{y}}{\left(\frac{2+\sqrt{4-y^2}}{y}\right)^2 + 1}$$
$$= \frac{\frac{8+4\sqrt{4-y^2}}{y}}{\frac{(2+\sqrt{4-y^2})^2+y^2}{y^2}} = \frac{8+4\sqrt{4-y^2}}{y} \cdot \frac{y^2}{(2+\sqrt{4-y^2})^2+y^2}$$
$$= \frac{y(8+4\sqrt{4-y^2})}{(2+\sqrt{4-y^2})^2+y^2}.$$

Computing the denominator,

$$(2+\sqrt{4-y^2})^2 + y^2 = 4 + 4\sqrt{4-y^2} + (4-y^2) + y^2 = 8 + 4\sqrt{4-y^2}.$$

Thus $(f \circ f^{-1})(y) = y$.

Exercise 3.3:

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{x}{x^2+1}.$$

(1) Compute $f(2)$ and $f\left(\frac{1}{2}\right)$.

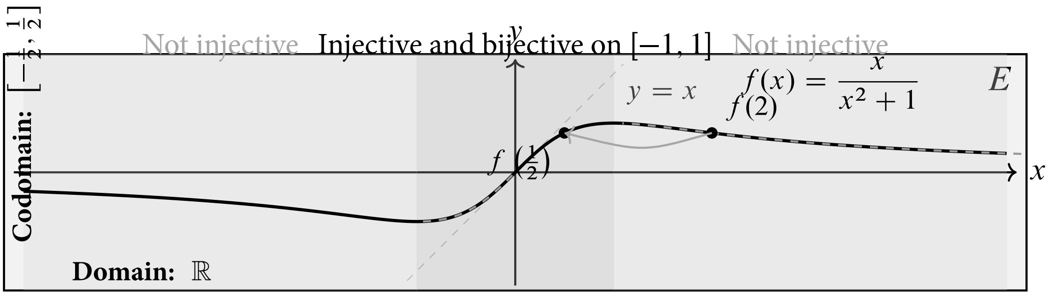
(2) Is f injective? Justify your answer.

(3) Find all real solutions to $f(x) = 2$. Is f surjective?

(4) Determine the image $f(\mathbb{R})$.

(Indication: using $x^2 + 1 \geq 0$)

Solution:



1. **Function Evaluation**

$$f(2) = \frac{2}{2^2+1} = \frac{2}{4+1} = \frac{2}{5},$$

$$f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2+1} = \frac{\frac{1}{2}}{\frac{1}{4}+1} = \frac{\frac{1}{2}}{\frac{5}{4}} = \frac{1}{2}\cdot\frac{4}{5} = \frac{2}{5}.$$

Thus,

$$\boxed{f(2) = \frac{2}{5},} \text{ and } \boxed{f\left(\frac{1}{2}\right) = \frac{2}{5}.}$$

2. **Injectivity**

From the computations above, $f(2) = f\left(\frac{1}{2}\right) = \frac{2}{5}$ but $2 \neq \frac{1}{2}$. This provides a counterexample to injectivity.

General analysis: Let $f(x_1) = f(x_2)$,

$$\begin{aligned} \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1} &\iff x_1(x_2^2+1) = x_2(x_1^2+1) \\ &\iff x_1x_2^2 + x_1 = x_1^2x_2 + x_2 \\ &\iff x_1x_2^2 - x_1^2x_2 + x_1 - x_2 = 0 \\ &\iff x_1x_2(x_2 - x_1) - (x_2 - x_1) = 0 \\ &\iff (x_2 - x_1)(x_1x_2 - 1) = 0. \end{aligned}$$

Thus $x_1 = x_2$ or $x_1x_2 = 1$, showing

$$\boxed{f \text{ is not injective.}}$$

3. **Solution of $f(x) = 2$ and Surjectivity**

   Solve $f(x) = 2$,

$$\begin{aligned} f(x) = 2 &\iff \frac{x}{x^2+1} = 2 \\ &\iff x = 2(x^2+1) \\ &\iff x = 2x^2 + 2 \\ &\iff 2x^2 - x + 2 = 0. \end{aligned}$$

   The discriminant is,

$$\Delta = (-1)^2 - 4 \cdot 2 \cdot 2 = 1 - 16 = -15 < 0 \implies \text{No real solutions exist.}$$

   Therefore, $2 \notin f(\mathbb{R})$, so

$$\boxed{f \text{ is not surjective on to } \mathbb{R}.}$$

4. **Image $f(\mathbb{R})$**

   Observe that, for any $x \in \mathbb{R}$,

$$\begin{aligned} (|x|-1)^2 &= |x|^2 - 2|x| + 1 \geq 0 \\ &\implies |x|^2 + 1 \geq 2|x| \\ &\implies |f(x)| = \frac{|x|}{x^2+1} \leq \frac{1}{2} \\ &\iff -\frac{1}{2} \leq f(x) = \frac{x}{x^2+1} \leq \frac{1}{2}, \end{aligned}$$

   with equality when $|x| = 1$.

   Therefore, the image is,

$$\boxed{f(\mathbb{R}) = \left[-\frac{1}{2}, \frac{1}{2}\right].}$$

   Alternatively, let $y \in \mathbb{R}$. Solve $f(x) = y$,

$$\begin{aligned} f(x) = y &\iff \frac{x}{x^2+1} = y \\ &\iff x = y(x^2+1) \\ &\iff yx^2 - x + y = 0 \quad (1). \end{aligned}$$

   Case 1: $y = 0$ Equation (1) becomes $-x = 0 \Rightarrow x = 0$. So $y = 0 \in f(\mathbb{R})$.

   Case 2: $y \neq 0$ For real solutions, discriminant must be non-negative,

$$\Delta = (-1)^2 - 4 \cdot y \cdot y = 1 - 4y^2 \geq 0 \iff y^2 \leq \frac{1}{4} \iff |y| \leq \frac{1}{2}.$$

   Verification of endpoints,

   - $y = \frac{1}{2}$: $\frac{1}{2}x^2 - x + \frac{1}{2} = 0 \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow x = 1$

- $y = -\frac{1}{2}$: $-\frac{1}{2}x^2 - x - \frac{1}{2} = 0 \Rightarrow x^2 + 2x + 1 = 0 \Rightarrow x = -1$

Exercise 3.4:
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^3 - 3x^2 - 4x + 12.$$

(a) Let $A = \{2, 0, -2, 1\}$ and $B = \{12\}$.

(1) Determine the set $f(A)$.

(2) Deduce that f is not injective. Justify this conclusion.

(3) Determine the preimage $f^{-1}(B)$.

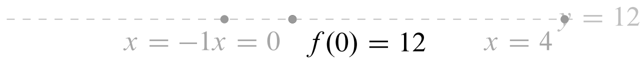
(b) Let R be the binary relation defined by

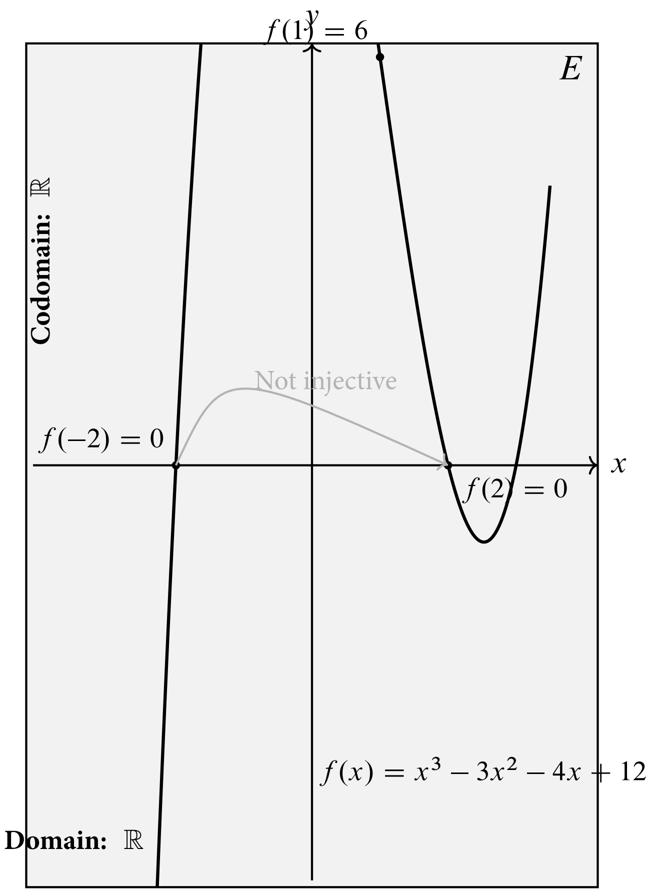
$$\forall x, y \in \mathbb{R}, \quad xRy \iff f(x) = f(y).$$

(1) Prove that R is an equivalence relation.

(2) Determine the equivalence classes of -2, 0, and 2.

Solution 3.1:





Part (a).

1. **Determination of $f(A)$ for $A = \{2, 0, -2, 1\}$**

$$f(2) = 8 - 12 - 8 + 12 = 0,$$
$$f(0) = 0 - 0 - 0 + 12 = 12,$$
$$f(-2) = -8 - 12 + 8 + 12 = 0,$$
$$f(1) = 1 - 3 - 4 + 12 = 6.$$

Thus,

$$\boxed{f(A) = \{0, 6, 12\}.}$$

2. **Non-injectivity Deduction**

$f(2) = f(-2) = 0$ but $2 \neq -2$, so f is **not injective**.

3. **Preimage $f^{-1}(B)$ for $B = \{12\}$**

Solve $f(x) = 12$,

$$\begin{aligned} x^3 - 3x^2 - 4x + 12 = 12 &\iff x^3 - 3x^2 - 4x = 0 \\ &\iff x(x^2 - 3x - 4) = 0 \\ &\iff x(x-4)(x+1) = 0. \end{aligned}$$

Thus,

$$\boxed{f^{-1}(\{12\}) = \{-1, 0, 4\}.}$$

Part (b).

1. **Equivalence Relation**
   - **Reflexivity**: $\forall x \in \mathbb{R}$, $f(x) = f(x)$, hence xRx.
   - **Symmetry**: If xRy, then $f(x) = f(y)$, so $f(y) = f(x)$, hence yRx.
   - **Transitivity**: If xRy and yRz, then $f(x) = f(y)$ and $f(y) = f(z)$, so $f(x) = f(z)$, hence xRz.

   Thus,

   $$\boxed{R \text{ is an equivalence relation.}}$$

2. **Equivalence Classes**
   - $[-2] = \{x \in \mathbb{R} \mid f(x) = f(-2) = 0\}$

     Factor $f(x)$,

     $$\begin{aligned} f(x) &= x^3 - 3x^2 - 4x + 12 \\ &= (x-2)(ax^2 + bx + c) \quad (\text{since } f(2) = 0) \\ &= ax^3 + (b-2a)x^2 + (c-2b)x - 2c \\ &= (x-2)(x^2 - x - 6) \quad (\text{since } a = 1, b - 2a = -3, \text{ and } -2c = 12) \\ &= (x-2)(x-3)(x+2). \end{aligned}$$

     Thus $f(x) = 0$ when $x = -2, 2, 3$, so

     $$\boxed{[-2] = \{-2, 2, 3\}.}$$

   - $[0] = \{x \in \mathbb{R} \mid f(x) = f(0) = 12\}$

     From part (a), $f(x) = 12$ when $x = -1, 0, 4$, so

     $$\boxed{[0] = \{-1, 0, 4\}.}$$

   - $[2] = \{x \in \mathbb{R} \mid f(x) = f(2) = 0\}$

     From above, $f(x) = 0$ when $x = -2, 2, 3$, so

     $$\boxed{[2] = \{-2, 2, 3\}.}$$

Note that $[-2] = [2]$ and $[0]$ is distinct.

4. Solutions of Tutorial Exercises (4):

A. LESLOUS A. AYACHI

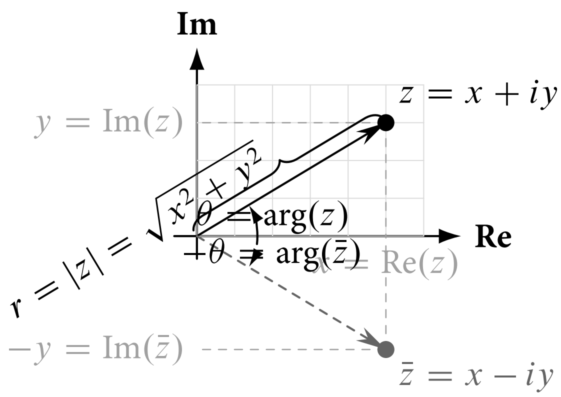
Theoretical Framework

Definition 4.1 (Complex Number Definition):
A complex number is an ordered pair of real numbers expressed in canonical form,

$$z = x + iy$$

where we have the fundamental properties,

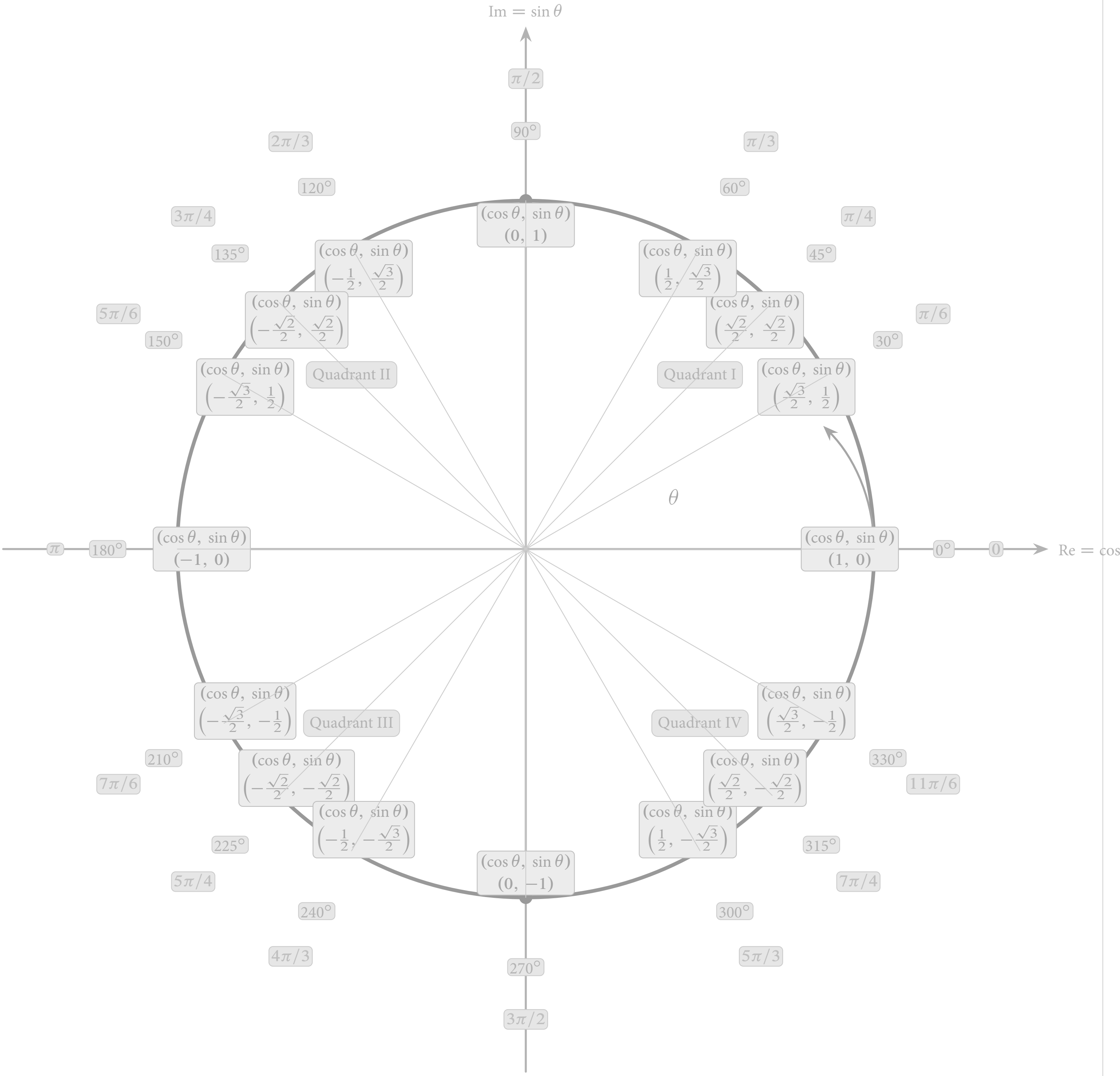
$$x = \mathrm{Re}(z) \quad \text{(real part)},$$
$$y = \mathrm{Im}(z) \quad \text{(imaginary part)},$$
$$i^2 = -1 \quad \text{(imaginary unit definition)}.$$



The complex conjugate exhibits fundamental symmetry,

$$\bar{z} = \overline{x + iy} = x - iy.$$

Definition 4.2 (Modulus and Principal Argument):
For $z = x + iy \in \mathbb{C}$, we define the modulus and argument as follows,



$$|z| = r = \sqrt{x^2 + y^2} \quad \text{(Modulus).}$$

$$\arg(z) = \begin{cases} \arctan\left(\frac{y}{x}\right) & x > 0 \quad \text{(Quadrants I/IV)}, \\ \arctan\left(\frac{y}{x}\right) + \pi & x < 0, y \geq 0 \quad \text{(Quadrant II)}, \\ \arctan\left(\frac{y}{x}\right) - \pi & x < 0, y < 0 \quad \text{(Quadrant III)}, \\ \frac{\pi}{2} & x = 0, y > 0, \\ -\frac{\pi}{2} & x = 0, y < 0, \\ \text{undefined} & x = 0, y = 0. \end{cases}$$

Theorem 4.1 (Polar Form Representation):
Every nonzero complex number admits a unique polar representation,

$$z = r(\cos\theta + i \sin\theta),$$

where we have the fundamental relationships,

$$r = |z| \geq 0, \quad \theta = \arg(z) \in (-\pi, \pi].$$
$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad \tan\theta = \frac{y}{x}.$$

Theorem 4.2 (Euler's Formula - Fundamental Identity):
The exponential function exhibits profound connection with trigonometry,

$$e^{i\theta} = \cos\theta + i \sin\theta, \quad z = re^{i\theta} = |z|e^{i \arg(z)}.$$

Comprehensive Trigonometric Framework

Identity 1: Pythagorean Foundation,

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta. \end{aligned}$$

Identity 2: Angle Composition Laws,

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\ \tan(\alpha \pm \beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}. \end{aligned}$$

Identity 3: Multiple Angle Relations,

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta. \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta. \end{aligned}$$

Identity 4: Power Reduction Formulas,

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}, \quad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}.$$

Theorem 4.3 (De Moivre's Theorem - Fundamental Power Law):
For any integer $k \in \mathbb{Z}$ and $\theta \in \mathbb{R}$, we have,

$$(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta), \quad (re^{i\theta})^k = r^k e^{ik\theta}.$$

Theorem 4.4 (Complex Arithmetic in Polar Coordinates):
For $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$ with $z_2 \neq 0$, we obtain,

Multiplication: $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$.

Modulus Properties: $|z_1 z_2| = |z_1||z_2|, \quad \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

Argument Properties: $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2), \quad \arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

$\arg(\overline{z}) = -\arg(z), \quad \arg\left(z^k\right) = k\arg(z)$.

Exercise 4.1 (Modulus and Argument Computation):
Compute the modulus and principal argument for,

$$z_1 = 1 + i, \quad z_2 = 1 - i, \quad z_3 = 1 + i\sqrt{3}, \quad z_4 = \frac{1 + i\sqrt{3}}{1 - i}.$$

Solution:

Case 1: For $z_1 = 1 + i$,

We begin by identifying the real and imaginary components,

$$x_1 = 1, \quad y_1 = 1.$$

Then we compute the modulus,

$$r_1 = \sqrt{x_1^2 + y_1^2} = \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}.$$

Next, we determine the trigonometric ratios,

$$\cos\theta_1 = \frac{x_1}{r_1} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \sin\theta_1 = \frac{y_1}{r_1} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Now we analyze the quadrant,

$$\cos\theta_1 > 0, \sin\theta_1 > 0 \Rightarrow \text{First quadrant.}$$

We find the reference angle,

$$\theta_{\text{ref}} = \frac{\pi}{4} \quad (\text{since } \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}).$$

Therefore, the principal argument is,

$$\theta_1 = \theta_{\text{ref}} = \frac{\pi}{4}.$$

Hence,

$$\boxed{|z_1| = \sqrt{2}, \quad \arg(z_1) = \frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}.}$$

Case 2: For $z_2 = 1 - i$,

We identify the components,

$$x_2 = 1, \quad y_2 = -1.$$

Then we compute the modulus,

$$r_2 = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}.$$

Next, we perform trigonometric analysis,

$$\cos\theta_2 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \sin\theta_2 = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

Now we determine the quadrant,

$$\cos\theta_2 > 0, \sin\theta_2 < 0 \Rightarrow \text{Fourth quadrant.}$$

We find the reference angle,

$$\theta_{\text{ref}} = \frac{\pi}{4}.$$

Therefore, the principal argument is,

$$\theta_2 = -\theta_{\text{ref}} = -\frac{\pi}{4}.$$

Hence,

$$\boxed{|z_2| = \sqrt{2}, \quad \arg(z_2) = -\frac{\pi}{4} + 2k\pi \equiv \underbrace{-\frac{\pi}{4} + 2\pi}_{=\frac{7\pi}{4}} + 2k\pi,\ k \in \mathbb{Z}.}$$

Method 2:

We have,

$$z_2 = \overline{z_1} \Longrightarrow \arg(z_2) = \arg(\overline{z_1}) = -\arg(z_1) = -\frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}.$$

Case 3: For $z_3 = 1 + i\sqrt{3}$,

We identify the components,

$$x_3 = 1, \quad y_3 = \sqrt{3}.$$

Then we compute the modulus,

$$r_3 = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2.$$

Next, we perform trigonometric analysis,

$$\cos\theta_3 = \frac{1}{2}, \quad \sin\theta_3 = \frac{\sqrt{3}}{2}.$$

Now we determine the quadrant,

$$\cos\theta_3 > 0, \sin\theta_3 > 0 \Rightarrow \text{First quadrant.}$$

We find the reference angle,

$$\theta_{\text{ref}} = \frac{\pi}{3} \quad (\text{since } \cos\frac{\pi}{3} = \frac{1}{2}, \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}).$$

Therefore, the principal argument is,

$$\theta_3 = \theta_{\text{ref}} = \frac{\pi}{3}.$$

Hence,

$$\boxed{|z_3| = 2, \quad \arg(z_3) = \frac{\pi}{3} + 2k\pi, \; k \in \mathbb{Z}.}$$

Case 4: For $z_4 = \dfrac{1+i\sqrt{3}}{1-i}$**,**

We begin with modulus computation,

$$|z_4| = \frac{|1+i\sqrt{3}|}{|1-i|} = \frac{\sqrt{1^2+(\sqrt{3})^2}}{\sqrt{1^2+(-1)^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Then we compute the argument,

$$\begin{aligned}\arg(z_4) &= \arg(1+i\sqrt{3}) - \arg(1-i) = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{3} + \frac{\pi}{4} \\ &= \frac{4\pi + 3\pi}{12} = \frac{7\pi}{12}.\end{aligned}$$

Therefore,

$$\boxed{|z_4| = \sqrt{2}, \quad \arg(z_4) = \frac{7\pi}{12} + 2k\pi, \; k \in \mathbb{Z}.}$$

□

Exercise 4.2 (Algebraic Form Transformation):
Express each complex number in canonical form $x + iy$,

(a) $\sqrt{2} - i - i(1 - i\sqrt{2})$,

(b) $(2 - 3i)(-2 + i)$,

(c) $(1 - i)(2 - i)(3 - i)$,

(d) $\dfrac{4 + 3i}{3 - 4i}$,

(e) $\dfrac{1 + i}{i} + \dfrac{i}{1 - i}$,

(f) $\dfrac{1 + 2i}{3 - 4i} + \dfrac{2 - i}{5i}$,

(g) $(1 - i\sqrt{3})^{-10}$,

(h) $(-1 + i)^7$.

Prove the identity,

$$\frac{(1 - i)^{46}(\cos\frac{\pi}{40} + i\sin\frac{\pi}{40})^{10}}{(8i - 8\sqrt{3})^6} = \frac{1}{4}(\sqrt{2} - i\sqrt{2}).$$

Solution:

Problem a: Simplification: $\sqrt{2} - i - i(1 - i\sqrt{2})$

We have,

$$\begin{aligned} z &= \sqrt{2} - i - i(1 - i\sqrt{2}) \\ &= \sqrt{2} - i - i + i^2\sqrt{2} \\ &= \sqrt{2} - 2i - \sqrt{2} \text{ (using } i^2 = -1) \\ &= -2i. \end{aligned}$$

Hence, we obtain,

$$\boxed{z = 0 - 2i.}$$

Problem b: Product expansion: $(2 - 3i)(-2 + i)$

We have,

$$z = (2 - 3i)(-2 + i)$$

$$
\begin{aligned}
&= 2(-2) + 2(i) + (-3i)(-2) + (-3i)(i) \\
&= -4 + 2i + 6i - 3i^2 \\
&= -4 + 8i + 3 \text{ (using } i^2 = -1).
\end{aligned}
$$

Hence, we obtain,

$$\boxed{z = -1 + 8i.}$$

Problem c: Sequential multiplication: $(1-i)(2-i)(3-i)$

We begin with the first multiplication,

$$
\begin{aligned}
(1-i)(2-i) &= 1 \cdot 2 + 1 \cdot (-i) + (-i) \cdot 2 + (-i)(-i) \\
&= 2 - i - 2i + i^2 \\
&= 2 - 3i - 1 \text{ (using } i^2 = -1) \\
&= 1 - 3i.
\end{aligned}
$$

Next, we multiply by the third factor,

$$
\begin{aligned}
(1-3i)(3-i) &= 1 \cdot 3 + 1 \cdot (-i) + (-3i) \cdot 3 + (-3i)(-i) \\
&= 3 - i - 9i + 3i^2 \\
&= 3 - 10i - 3 \text{ (using } i^2 = -1).
\end{aligned}
$$

Therefore,

$$\boxed{z = 0 - 10i.}$$

Problem d: Rationalization: $\dfrac{4+3i}{3-4i}$

We multiply numerator and denominator by the conjugate,

$$
\begin{aligned}
z &= \frac{(4+3i)(3+4i)}{(3-4i)(3+4i)} \\
&= \frac{4 \cdot 3 + 4 \cdot 4i + 3i \cdot 3 + 3i \cdot 4i}{9+16} \\
&= \frac{12 + 16i + 9i + 12i^2}{25} \\
&= \frac{12 + 25i - 12}{25} = \frac{25i}{25} \text{ (using } i^2 = -1).
\end{aligned}
$$

Therefore,

$$\boxed{z = 0 + i.}$$

Problem e: Sum of fractions: $\dfrac{1+i}{i} + \dfrac{i}{1-i}$

We begin by simplifying the first term,

$$\frac{1+i}{i} = \frac{1}{i} + \frac{i}{i} = -i + 1 = 1 - i.$$

Then we simplify the second term,

$$\frac{i}{1-i} = \frac{i(1+i)}{(1-i)(1+i)} = \frac{i+i^2}{1+1} = \frac{i-1}{2} = -\frac{1}{2} + \frac{1}{2}i.$$

Now we compute the sum,

$$(1-i) + \left(-\frac{1}{2} + \frac{1}{2}i\right) = \left(1 - \frac{1}{2}\right) + \left(-1 + \frac{1}{2}\right)i = \frac{1}{2} - \frac{1}{2}i.$$

Therefore,

$$\boxed{z = \frac{1}{2} - \frac{1}{2}i.}$$

Problem f: Complex sum: $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$

We begin by simplifying the first term,

$$\begin{aligned}\frac{1+2i}{3-4i} &= \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} = \frac{1\cdot 3 + 1\cdot 4i + 2i\cdot 3 + 2i\cdot 4i}{9+16} \\ &= \frac{3+4i+6i+8i^2}{25} = \frac{3+10i-8}{25} = \frac{-5+10i}{25} = -\frac{1}{5} + \frac{2}{5}i.\end{aligned}$$

Next, we simplify the second term,

$$\frac{2-i}{5i} = \frac{(2-i)(-i)}{5i(-i)} = \frac{-2i+i^2}{5(-i^2)} = \frac{-2i-1}{5} = -\frac{1}{5} - \frac{2}{5}i.$$

Now we compute the sum,

$$\left(-\frac{1}{5} + \frac{2}{5}i\right) + \left(-\frac{1}{5} - \frac{2}{5}i\right) = -\frac{2}{5}.$$

Therefore, we obtain,

$$\boxed{z = -\frac{2}{5} + 0i.}$$

Problem g: Negative power: $(1 - i\sqrt{3})^{-10}$

We begin by converting to polar form.

For $1 - i\sqrt{3}$, we have

$$r = \sqrt{(1)^2 + \sqrt{3}^2} = 2, \ \cos(\theta) = \frac{1}{2} > 0, \ \sin(\theta) = -\frac{\sqrt{3}}{2} < 0$$

$$\implies \text{Fourth quadrant} \implies \theta_{\text{ref}} = \frac{\pi}{3} \implies \theta = -\theta_{\text{ref}} = -\frac{\pi}{3}.$$

Then

$$1 - i\sqrt{3} = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 2e^{-i\pi/3}.$$

Then we compute the power,

$$(1 - i\sqrt{3})^{-10} = 2^{-10}e^{i10\pi/3} = \frac{1}{1024}e^{i10\pi/3}.$$

Now we reduce the angle,

$$e^{i10\pi/3} = e^{i(6\pi+4\pi)/3} = e^{i4\pi/3} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Hence, we have,

$$\boxed{z = -\frac{1}{2048} - i\frac{\sqrt{3}}{2048}.}$$

Problem h: Seventh power: $(-1 + i)^7$

We begin by converting to polar form.

For $-1 + i$, we have

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}, \ \cos(\theta) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} < 0, \ \sin(\theta) = \frac{\sqrt{2}}{2}$$

$$\implies \text{Second quadrant} \implies \theta_{\text{ref}} = \frac{\pi}{4} \implies \theta = \pi - \theta_{\text{ref}} = \frac{3\pi}{4}.$$

Then

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}e^{i3\pi/4}.$$

Then we compute the power,

$$(-1 + i)^7 = (\sqrt{2})^7 e^{i21\pi/4} = 2^{7/2}e^{i21\pi/4}.$$

Now we reduce the angle,

$$e^{i21\pi/4} = e^{i(5\pi+\pi/4)} = e^{i5\pi}e^{i\pi/4} = (-1)e^{i\pi/4} = -e^{i\pi/4}$$
$$= -\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = -\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right).$$

Hence, we have,

$$\boxed{z = -8 - 8i.}$$

Problem i: Identity proof:

We begin with the left side,

$$L = \frac{(1-i)^{46}(\cos\frac{\pi}{40} + i\sin\frac{\pi}{40})^{10}}{(8i - 8\sqrt{3})^6}.$$

First, we analyze $(1-i)^{46}$,

For $1-i$, we have

$$\begin{aligned} r &= \sqrt{(1)^2 + (-1)^2} = \sqrt{2},\ \cos(\theta) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} > 0,\ \sin(\theta) = -\frac{\sqrt{2}}{2} < 0 \\ &\implies \text{Fourth quadrant} \implies \theta_{\text{ref}} = \frac{\pi}{4} \implies \theta = -\theta_{\text{ref}} = -\frac{\pi}{4}. \end{aligned}$$

Then,

$$1 - i = \sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right) = \sqrt{2}e^{-i\pi/4}$$

$$\begin{aligned} \implies (1-i)^{46} &= (\sqrt{2})^{46} e^{-i46\pi/4} = 2^{23} e^{-i(11\pi + 2\pi/4)} \\ &= 2^{23} e^{-i11\pi} e^{-i\pi/2} \\ &= 2^{23} \underbrace{(\cos(-11\pi) + i\sin(-11\pi))}_{=-1} \underbrace{(\cos(-\pi/2) + i\sin(-\pi/2))}_{=-i} \\ &= 2^{23} \cdot i. \end{aligned}$$

Next, we analyze $(8i - 8\sqrt{3})^6$,

For $i - \sqrt{3}$, we have

$$\begin{aligned} r &= \sqrt{(\sqrt{3})^2 + (1)^2} = 2,\ \cos(\theta) = -\frac{\sqrt{3}}{2} < 0,\ \sin(\theta) = \frac{1}{2} > 0 \\ &\implies \text{Second quadrant} \implies \theta_{\text{ref}} = \frac{\pi}{6} \implies \theta = \pi - \theta_{\text{ref}} = \frac{5\pi}{6}. \end{aligned}$$

Then

$$8i - 8\sqrt{3} = 8(i - \sqrt{3}) = 8\left(2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right) = 16e^{i5\pi/6},$$

$$\implies (8i - 8\sqrt{3})^6 = 16^6 e^{i30\pi/6} = 2^{24} e^{i5\pi} = -2^{24},$$

and

$$\cos\frac{\pi}{40} + i\sin\frac{\pi}{40} = e^{i\pi/40}.$$

Now we combine the results,

$$L = \frac{2^{23} \cdot i.(e^{i\pi/40})^{10}}{-2^{24}} = -\frac{1}{2} i\, e^{i\pi/4} \underset{?}{=} -\frac{1}{2} i\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= \frac{1}{4}(\sqrt{2} - i\sqrt{2}).$$

Therefore,

$$\boxed{\frac{(1-i)^{46}(\cos\frac{\pi}{40} + i\sin\frac{\pi}{40})^{10}}{(8i - 8\sqrt{3})^6} = \frac{1}{4}(\sqrt{2} - i\sqrt{2}).}$$

□

Exercise 4.3 (Real and Imaginary Parts Analysis):
For $z = x + iy$, express in terms of real and imaginary parts,

(a) z^3,

(b) $z\bar{z}$,

(c) $\frac{1}{z}$ $(z \neq 0)$,

(d) $\frac{z-i}{1-\bar{z}i}$ $(z \neq i)$,

(e) $|z-i|^2$,

(f) $|z|^4$,

(g) $\frac{|z+1|}{|z-1|}$ $(z \neq 1)$.

Solution:

Part a: Cube expansion: z^3

We have,

$$\begin{aligned} z^3 &= (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\ &= x^3 + 3ix^2y + 3x(i^2y^2) + i^3y^3 \\ &= x^3 + 3ix^2y - 3xy^2 - iy^3 \text{ (using } i^2 = -1 \text{ and } i^3 = i^2 i = -i) \\ &= (x^3 - 3xy^2) + i(3x^2y - y^3). \end{aligned}$$

Hence,

$$\boxed{\text{Re}(z^3) = x^3 - 3xy^2, \quad \text{Im}(z^3) = 3x^2y - y^3.}$$

Part b: Conjugate product: $z\bar{z}$

We compute the product,

$$\begin{aligned} z\bar{z} &= (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 - i^2y^2 \\ &= x^2 + y^2 \,(\text{using}\, i^2 = -1). \end{aligned}$$

Therefore,

$$\boxed{\operatorname{Re}(z\bar{z}) = x^2 + y^2, \quad \operatorname{Im}(z\bar{z}) = 0.}$$

Part c: Reciprocal: $\frac{1}{z}$

We begin with rationalization,

$$\begin{aligned} \frac{1}{z} &= \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}. \end{aligned}$$

Therefore,

$$\boxed{\operatorname{Re}\left(\frac{1}{z}\right) = \frac{x}{x^2+y^2}, \quad \operatorname{Im}\left(\frac{1}{z}\right) = -\frac{y}{x^2+y^2}.}$$

Part d: Rational function: $\frac{z-i}{1-\bar{z}i}$

We begin by expressing numerator and denominator,

$$\begin{aligned} \text{Numerator:} \quad & z - i = x + i(y-1), \\ \text{Denominator:} \quad & 1 - \bar{z}i = 1 - (x-iy)i = 1 - (xi - i^2y) = 1 - y - xi. \end{aligned}$$

Then we form the expression,

$$\begin{aligned} \frac{z-i}{1-\bar{z}i} &= \frac{x+i(y-1)}{1-y-xi} \\ &= \frac{[x+i(y-1)][1-y+xi]}{(1-y)^2+x^2} \text{ (using the conjugate)} \\ &= \frac{x(1-y)+ix^2+i(y-1)(1-y)+i^2x(y-1)}{x^2+(1-y)^2} \\ &= \frac{x(1-y)+ix^2-i(y-1)^2-x(y-1)}{x^2+(1-y)^2} \text{ (using } i^2=-1) \\ &= \frac{2x(1-y)+i[x^2-(y-1)^2]}{x^2+(1-y)^2}. \end{aligned}$$

Therefore,

$$\boxed{\operatorname{Re}\left(\frac{z-i}{1-\bar{z}i}\right)=\frac{2x(1-y)}{x^2+(1-y)^2},\quad \operatorname{Im}\left(\frac{z-i}{1-\bar{z}i}\right)=\frac{x^2-(y-1)^2}{x^2+(1-y)^2}.}$$

Part e: Distance squared: $|z-i|^2$

We compute,

$$|z-i|^2=|x+i(y-1)|^2=x^2+(y-1)^2$$

Therefore,

$$\boxed{\operatorname{Re}|z-i|^2=x^2+(y-1)^2,\quad \operatorname{Im}|z-i|^2=0.}$$

Part f: Fourth power modulus: $|z|^4$

We compute,

$$|z|^4=(|z|^2)^2=(x^2+y^2)^2.$$

Therefore,

$$\boxed{\operatorname{Re}|z|^4=(x^2+y^2)^2,\quad \operatorname{Im}|z|^4=0.}$$

Part g: Ratio of distances: $\dfrac{|z+1|}{|z-1|}$

We compute,

$$\frac{|z+1|}{|z-1|}=\frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}}.$$

Therefore,

$$\boxed{\operatorname{Re}\frac{|z+1|}{|z-1|}=\frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}},\quad \operatorname{Im}\frac{|z+1|}{|z-1|}=0.}$$

□

Exercise 4.4 (Polar Form Transformation):
Express in polar form $r(\cos\theta+i\sin\theta)$ with $-\pi\le\theta\le\pi$,

$$1+i,\quad 1+i\sqrt{3},\quad 1-i\sqrt{3},\quad -5,\quad \frac{(1-i)(\sqrt{3}+i)}{(1+i)(\sqrt{3}-i)},\quad -8+\frac{4}{i}+\frac{25}{3-4i}.$$

Solution:

Number 1: **For $z = 1 + i$,**

We compute the modulus,

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Then we find the trigonometric ratios,

$$\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Since both cosine and sine are positive, we have,

$$\theta = \frac{\pi}{4} \quad \text{(First quadrant)}$$

Therefore,

$$\boxed{z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\,\sin\frac{\pi}{4}\right).}$$

Number 2: **For $z = 1 + i\sqrt{3}$,**

We compute the modulus,

$$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2.$$

Then we find the trigonometric ratios,

$$\cos\theta = \frac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2}.$$

Since both cosine and sine are positive, we have,

$$\theta = \frac{\pi}{3} \quad \text{(First quadrant).}$$

Therefore,

$$\boxed{z = 2\left(\cos\frac{\pi}{3} + i\,\sin\frac{\pi}{3}\right).}$$

Number 3: **For $z = 1 - i\sqrt{3}$,**

We compute the modulus,

$$r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2.$$

Then we find the trigonometric ratios,

$$\cos\theta = \frac{1}{2}, \quad \sin\theta = -\frac{\sqrt{3}}{2}.$$

Since cosine is positive and sine is negative, we have,

$$\theta = -\frac{\pi}{3} \quad \text{(Fourth quadrant).}$$

Therefore,

$$\boxed{z = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right).}$$

Number 4: For $z = -5$,

We compute the modulus,

$$r = 5.$$

Then we find the trigonometric ratios,

$$\cos\theta = -1, \quad \sin\theta = 0.$$

Therefore, we have,

$$\theta = \pi.$$

Hence,

$$\boxed{z = 5(\cos\pi + i\sin\pi).}$$

Number 5: For $z = \dfrac{(1-i)(\sqrt{3}+i)}{(1+i)(\sqrt{3}-i)}$,

We compute the modulus,

$$r = \frac{|1-i||\sqrt{3}+i|}{|1+i||\sqrt{3}-i|} = \frac{\sqrt{2}\cdot 2}{\sqrt{2}\cdot 2} = 1.$$

Then we compute the argument,

$$\begin{aligned}\theta &= \arg(\sqrt{3}+i) - \arg(\sqrt{3}-i) + \arg(1-i) - \arg(1+i)\\ &= 2\arg(\sqrt{3}+i) - 2\arg(1+i) = 2\cdot\frac{\pi}{6} - 2\cdot\frac{\pi}{4} = -\frac{\pi}{6}.\end{aligned}$$

Therefore,

$$\boxed{z = \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right).}$$

Number 6: For $z = -8 + \dfrac{4}{i} + \dfrac{25}{3-4i}$,

We simplify each term,

$$\frac{4}{i} = -4i, \quad \frac{25}{3-4i} = \frac{25(3+4i)}{9+16} = 3+4i.$$

Then we compute the sum,

$$-8-4i+3+4i=-5.$$

Therefore, we have,

$$r=5,\quad \theta=\pi.$$

Hence,

$$\boxed{z=5(\cos\pi+i\sin\pi).}$$

□

Exercise 4.5 (Complex Equations Solution):
Solve the equations,

(a) $z^3+3z^2+3z+3=0$

(b) $(z-1)^4=1$

Solution:

Equation a: Cubic equation: $z^3+3z^2+3z+3=0$

We observe that the left-hand side resembles a perfect cube,

$$z^3+3z^2+3z+1=(z+1)^3.$$

Therefore, we can rewrite the equation as,

$$(z+1)^3+2=0\Rightarrow(z+1)^3=-2.$$

Now we make the substitution,

$$w=z+1\Rightarrow w^3=-2.$$

Then we find the cube roots of -2,

$$w=\sqrt[3]{-2}=\sqrt[3]{2}\cdot\sqrt[3]{-1}=\sqrt[3]{2}\cdot\sqrt[3]{e^{i(\pi+2k\pi)}}=\sqrt[3]{2}\cdot e^{i(\pi+2k\pi)/3},\quad k=0,1,2.$$

So we have the explicit roots,

$$w_1=\sqrt[3]{2}e^{i\pi/3},\quad w_2=\sqrt[3]{2}e^{i\pi},\quad w_3=\sqrt[3]{2}e^{i5\pi/3}.$$

Now we substitute back to find z,

$$z=w-1.$$

Therefore, the solutions are,

$$\boxed{z_1=\sqrt[3]{2}e^{i\pi/3}-1=\sqrt[3]{2}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)-1}$$

$$\boxed{z_2 = \sqrt[3]{2}e^{i\pi} - 1 = -\sqrt[3]{2} - 1}$$

$$\boxed{z_3 = \sqrt[3]{2}e^{i5\pi/3} - 1 = \sqrt[3]{2}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) - 1}$$

Equation b: Quartic equation: $(z-1)^4 = 1$

We make the substitution,

$$w = z - 1 \Rightarrow w^4 = 1 = e^{i(0+2k\pi)}.$$

Then we find the fourth roots of unity,

$$w = e^{i\cdot 2\pi k/4} = e^{i\pi k/2}, \quad k = 0, 1, 2, 3.$$

So we have the explicit roots,

$$w_1 = e^{i\cdot 0} = 1, \quad w_2 = e^{i\pi/2} = i$$
$$w_3 = e^{i\pi} = -1, \quad w_4 = e^{i3\pi/2} = -i.$$

Now we substitute back to find z,

$$z = w + 1.$$

Therefore, the solutions are,

$$\boxed{z_1 = 1 + 1 = 2}, \quad \boxed{z_2 = 1 + i}, \quad \boxed{z_3 = 1 - 1 = 0}, \quad \boxed{z_4 = 1 - i.}$$

□

5. Homework (S1–S4) with Full Solutions

Exercise 5.1:
Let $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set, and consider,

$$A = \{1, 3, 5, 7, 9\}, \quad B = \{2, 3, 5, 7\}, \quad C = \{1, 4, 9\}.$$

1. Calculate the following sets,
   a) $A \cap B \cap C$
   b) $A \cup B \cup C$
   c) $C_E(A \cup B)$
   d) $(A \setminus B) \cap C$
   e) $A \triangle B$ (symmetric difference)
2. Verify the following properties,
   (a) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
   (b) $C_E(A \cap B) = C_E(A) \cup C_E(B)$

Solution:

1. Set Calculations
   a) $A \cap B \cap C$

      We compute,
      $$A \cap B = \{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 7\} = \{3, 5, 7\}.$$
      Then,
      $$(A \cap B) \cap C = \{3, 5, 7\} \cap \{1, 4, 9\} = \emptyset.$$
      Hence,
      $$\boxed{\emptyset}$$

   b) $A \cup B \cup C$

      We compute,
      $$A \cup B = \{1, 3, 5, 7, 9\} \cup \{2, 3, 5, 7\} = \{1, 2, 3, 5, 7, 9\}.$$
      Then,
      $$(A \cup B) \cup C = \{1, 2, 3, 5, 7, 9\} \cup \{1, 4, 9\} = \{1, 2, 3, 4, 5, 7, 9\}.$$

Hence,

$$\boxed{\{1,2,3,4,5,7,9\}}$$

c) $C_E(A \cup B)$

We have,

$$A \cup B = \{1,2,3,5,7,9\}, \quad \text{so} \quad C_E(A \cup B) = E \setminus (A \cup B) = \{4,6,8,10\}.$$

Hence,

$$\boxed{\{4,6,8,10\}}$$

d) $(A \setminus B) \cap C$

We compute,

$$A \setminus B = \{1,3,5,7,9\} \setminus \{2,3,5,7\} = \{1,9\}.$$

Then,

$$(A \setminus B) \cap C = \{1,9\} \cap \{1,4,9\} = \{1,9\}.$$

Hence,

$$\boxed{\{1,9\}}$$

e) $A \triangle B$ (symmetric difference)

Recall,

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

We have,

$$A \setminus B = \{1,9\}, \quad B \setminus A = \{2\}, \quad \text{so} \quad A \triangle B = \{1,9\} \cup \{2\} = \{1,2,9\}.$$

Hence,

$$\boxed{\{1,2,9\}}$$

2. Verification of Properties

a) Verify $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

First,

$$B \cap C = \{2,3,5,7\} \cap \{1,4,9\} = \emptyset,$$
$$\text{so} \quad A \setminus (B \cap C) = A \setminus \emptyset = A = \{1,3,5,7,9\}.$$

Now,

$$A \setminus B = \{1,9\}, \quad A \setminus C = \{3,5,7\},$$
$$\text{so} \quad (A \setminus B) \cup (A \setminus C) = \{1,9\} \cup \{3,5,7\} = \{1,3,5,7,9\}.$$

Since both sides equal $\{1,3,5,7,9\}$, the identity holds.

b) Verify $C_E(A \cap B) = C_E(A) \cup C_E(B)$.

First,

$$A \cap B = \{3, 5, 7\}, \quad \text{so} \quad C_E(A \cap B) = E \setminus \{3, 5, 7\} = \{1, 2, 4, 6, 8, 9, 10\}.$$

Now,

$$C_E(A) = E \setminus A = \{2, 4, 6, 8, 10\}, \quad C_E(B) = E \setminus B = \{1, 4, 6, 8, 9, 10\},$$
$$C_E(A) \cup C_E(B) = \{2, 4, 6, 8, 10\} \cup \{1, 4, 6, 8, 9, 10\} = \{1, 2, 4, 6, 8, 9, 10\}.$$

Since both sides equal $\{1, 2, 4, 6, 8, 9, 10\}$, the identity holds.

Exercise 5.2:
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by,

$$f(x) = x^4 - 6x^2 + 5.$$

Define a relation R on $\mathbb{R}$ by,

$$xRy \iff f(x) = f(y).$$

1. Show that R is an equivalence relation.
2. Solve the equation $f(x) = f(y)$ and find all possible relationships between x and y. *(Hint: factor $f(x) - f(y)$.)*
3. Determine the equivalence classes $[x]$.
   - For which values of x does the class contain more than one element?
4. Describe geometrically the equivalence classes in terms of the symmetries of the graph of $f(x)$.

Solution:

1. Equivalence relation.

   We verify the three properties:

   a) **Reflexivity:** For any $x \in \mathbb{R}$, we have $f(x) = f(x)$, so xRx. Hence,

   $$\boxed{R \text{ is reflexive.}}$$

   b) **Symmetry:** If xRy, then $f(x) = f(y)$, so $f(y) = f(x)$, hence yRx. Thus,

   $$\boxed{R \text{ is symmetric.}}$$

   c) **Transitivity:** If xRy and yRz, then $f(x) = f(y)$ and $f(y) = f(z)$, so $f(x) = f(z)$, hence xRz. Thus, R is transitive.

   $$\boxed{R \text{ is transitive.}}$$

   Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

2. Solutions of $f(x) = f(y)$ / Relationships between x and y

We compute,

$$f(x) - f(y) = (x^4 - 6x^2 + 5) - (y^4 - 6y^2 + 5) = x^4 - y^4 - 6(x^2 - y^2).$$

Factor,

$$x^4 - y^4 = (x^2 - y^2)(x^2 + y^2), \quad \text{so}$$
$$f(x) - f(y) = (x^2 - y^2)(x^2 + y^2) - 6(x^2 - y^2) = (x^2 - y^2)(x^2 + y^2 - 6).$$

Thus,

$$f(x) = f(y) \iff (x^2 - y^2)(x^2 + y^2 - 6) = 0.$$

This gives two cases,

$$\begin{array}{ll} \text{(i)} & x^2 - y^2 = 0 \implies x^2 = y^2 \implies x = y \quad \text{or} \quad x = -y, \\ \text{(ii)} & x^2 + y^2 - 6 = 0 \implies x^2 + y^2 = 6. \end{array}$$

Therefore, the solutions are,

$$\boxed{x = y \quad \text{or} \quad x = -y \quad \text{or} \quad x^2 + y^2 = 6}$$

3. Equivalence classes $[x]$.

The equivalence class of x is,

$$\begin{aligned} [x] &= \{y \in \mathbb{R} \mid f(x) = f(y)\} \\ &= \{y \in \mathbb{R} \mid x = y \text{ or } x = -y \text{ or } x^2 + y^2 = 6\}. \end{aligned}$$

- We analyze the structure of $[x]$ by considering different cases:
  - If $x^2 > 6$, then $x^2 + y^2 = 6$ has no real solution for y. So the only relations come from $x^2 = y^2$, so $y = x$ or $y = -x$. But note: if $x^2 > 6$, then $x \neq -x$ (unless $x = 0$, but $0^2 = 0 < 6$). So for $|x| > \sqrt{6}$, we have
$$[x] = \{x, -x\}.$$
  - If $x^2 = 6$, i.e., $x = \sqrt{6}$ or $x = -\sqrt{6}$, then,
$$\begin{aligned} f(x) &= 36 - 36 + 5 = 5, \\ f(y) = 5 &\Rightarrow y^4 - 6y^2 + 5 = 5 \\ &\Rightarrow y^4 - 6y^2 = 0 \Rightarrow y^2(y^2 - 6) = 0 \\ &\Rightarrow y = 0 \text{ or } y = \pm\sqrt{6}. \end{aligned}$$
    So
$$[x] = \{0, \sqrt{6}, -\sqrt{6}\}.$$
  - If $0 < |x| < \sqrt{6}$, then $x^2 + y^2 = 6$ gives $y = \pm\sqrt{6 - x^2}$. Also, $x^2 = y^2$ gives $y = \pm x$. So we have four potential distinct numbers,
$$x, -x, \sqrt{6 - x^2}, -\sqrt{6 - x^2}.$$
    However, note that if $x^2 = 3$, then $\sqrt{6 - x^2} = \sqrt{3} = |x|$, so there is overlap. Specifically:

    * If $x^2 = 3$, then $x = \pm\sqrt{3}$, and then

$$\sqrt{6-x^2} = \sqrt{3} = |x|,$$

      so the set becomes

$$\{\sqrt{3}, -\sqrt{3}\}.$$

    * If $x^2 \neq 3$, then these four numbers are distinct. So for $0 < |x| < \sqrt{6}$ and $|x| \neq \sqrt{3}$, we have

$$[x] = \{x, -x, \sqrt{6-x^2}, -\sqrt{6-x^2}\}.$$

  - If $x = 0$, then $f(0) = 5$. Then $f(y) = 5$ gives

$$y^4 - 6y^2 = 0 \Rightarrow y^2(y^2-6) = 0 \Rightarrow y = 0 \text{ or } y = \pm\sqrt{6}.$$

    So

$$[0] = \{0, \sqrt{6}, -\sqrt{6}\}.$$

   Now, for which x does the class contain more than one element? From the above, we see that for every $x \in \mathbb{R}$, the class $[x]$ has at least two elements. In fact,

   - If $|x| > \sqrt{6}$, then $[x] = \{x, -x\}$ (two elements).
   - If $|x| = \sqrt{6}$, then $[x] = \{0, \sqrt{6}, -\sqrt{6}\}$ (three elements).
   - If $0 < |x| < \sqrt{6}$ and $|x| \neq \sqrt{3}$, then $[x]$ has four elements.
   - If $|x| = \sqrt{3}$, then $[x] = \{\sqrt{3}, -\sqrt{3}\}$ (two elements).
   - If $x = 0$, then $[0]$ has three elements.

   So every equivalence class has at least two elements.

4. Geometric interpretation

   In fact, the equivalence relation identifies points that are symmetric with respect to the y-axis (since f is even) and also identifies points on the circle $x^2 + y^2 = 6$ (which is symmetric about both axes). The geometric interpretation is that the equivalence classes are the sets of points that are mapped to the same value by f, and these sets are symmetric under reflections about the y-axis and rotations by 180° about the origin (due to the $x^2 = y^2$ condition) and also include points on the circle of radius $\sqrt{6}$.

Exercise 5.3:

Let R be a relation on $\mathbb{R}$ defined by,

$$xRy \iff x^2 + y^2 = 10.$$

1. Determine whether R is,

   a) Reflexive

   b) Symmetric

   c) Antisymmetric

d) Transitive

2. Is R an equivalence relation or an order relation?

3. **Geometric interpretation:** Describe in the plane the set of all pairs (x, y) that satisfy xRy. *(Hint: what shape does the equation $x^2 + y^2 = 10$ represent?)*

Solution:

1. Determine properties

   a) **Reflexive:** R is reflexive if for every $x \in \mathbb{R}$, xRx, i.e.,

   $$x^2 + x^2 = 10 \implies 2x^2 = 10 \implies x^2 = 5 \implies x = \pm\sqrt{5}.$$

   This is not true for all x, so

   $$\boxed{R \text{ is not reflexive.}}$$

   b) **Symmetric:** If xRy, then $x^2 + y^2 = 10$, which is the same as $y^2 + x^2 = 10$, so yRx.

   Hence, R is symmetric.

   $$\boxed{R \text{ is symmetric.}}$$

   c) **Antisymmetric:** R is antisymmetric if xRy and yRx imply $x = y$.

   But consider

   $$x = 1, y = 3 : 1^2 + 3^2 = 1 + 9 = 10,$$

   so $1R3$ and $3R1$, but $1 \neq 3$. So

   $$\boxed{R \text{ is not antisymmetric.}}$$

   d) **Transitive:** R is transitive if xRy and yRz imply xRz.

   Take $x = 1$, $y = 3$, then $1R3$. Now we need yRz,

   $$3^2 + z^2 = 10 \implies z^2 = 1 \implies z = \pm 1.$$

   So if we take $z = 1$, then $3R1$. But then xRz would require $1^2 + 1^2 = 2 \neq 10$, so $1R1$.

   Hence,

   $$\boxed{R \text{ is not transitive.}}$$

2. Since R is not reflexive and not transitive, it is not an equivalence relation. It is also not an order relation because it is not antisymmetric and not transitive.

3. Geometric interpretation

   The condition $x^2 + y^2 = 10$ represents a circle in the plane centered at the origin with radius $\sqrt{10}$. So the relation R consists of all pairs (x, y) that lie on this circle. In other words, R is the set of points on the circle of radius $\sqrt{10}$.

6. Homework (S5—S9)

Exercise 6.1:
Let the universal set be $E = \mathbb{R} \setminus \{1\}$.
Define the following subsets of E,

$$A = \{x \in E \mid x^2 - 4x + 3 \leq 0\}, \quad B = \{x \in E \mid x^2 - 2x - 3 \geq 0\},$$
$$\text{and} \quad C = \{x \in E \mid |x - 2| \leq 2\}.$$

Let $f : E \to \mathbb{R}$ be the function

$$f(x) = \frac{x^2 - 4x + 3}{x - 1}.$$

(1) **Set reasoning**

a) Express A, B, and C as unions of intervals.

b) Determine $C_E(A)$, $A \cap B$, and $(A \cup C) \setminus B$.

c) Verify that,

$$C_E(A \cup B) = C_E(A) \cap C_E(B).$$

(2) **Domain and simplification of the function**

a) Simplify $f(x)$ algebraically when possible.

b) Determine the domain and points of discontinuity.

c) Compute the limits $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.

d) Sketch the qualitative behavior of $f(x)$ on E.

(3) **Functional properties**

a) Determine whether f is injective on E.

b) Find all real values y such that $f(x) = y$ has exactly two distinct real solutions.

c) Determine the image $f(\mathbb{R})$.

Exercise 6.2:
Define a binary relation R on E by,

$$xRy \iff f(x) = f(y).$$

(1) Prove that R is an equivalence relation on E.

(2) Determine the equivalence classes of $0, 2$, and 4.

(3) Define a new relation S on E by,

$$xSy \iff f(x) < f(y).$$

a) Show that S is a strict order relation.

b) Is S total on E? Justify.

(4) Let $g = f|_{[0,3]}$, the restriction of f to $[0, 3]$.

a) Show that g is bijective from $[0, 3]$ to its image.

b) Determine the inverse function $g^{-1}(x)$.

c) Verify that $f(g^{-1}(x)) = x$ for all x in the image of g.

(5) Let $T : E \to E$ be the composition $T(x) = f(f(x))$.

a) Compute and simplify $T(x)$.

b) Determine whether T is idempotent, i.e. $T(T(x)) = T(x)$.

c) Explain what this implies about the structure of R.

Exercise 6.3:

Let $\mathbb{C}$ be the set of complex numbers. For any $z = x + iy \in \mathbb{C}$, define,

$$g(z) = \frac{z-1}{z+1}.$$

(1) Find the domain of g and simplify $g(z)$ for $z = i, -i, 2i$, and $1 + i$.

(2) Show that $|g(z)| = 1$ if and only if $\mathrm{Re}(z) = 0$.

(3) Let R be the relation on $\mathbb{C} \setminus \{-1\}$ defined by,

$$z_1 R z_2 \iff g(z_1) = g(z_2).$$

Prove that R is an equivalence relation.

(4) Determine the equivalence class of $z = i$ under R.

(5) Let $h : \mathbb{C} \setminus \{-1\} \to \mathbb{C}$ be defined by

$$h(z) = \overline{g(z)}.$$

a) Show that $h(z) \cdot g(z) = 1$ if and only if $|z| = 1$.

b) Interpret geometrically the transformation g and its conjugate h in the complex plane.

c) Deduce that g maps the real axis to itself and the unit circle to the imaginary axis.

Solutions of Tutorial Exercises

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