University of 8 May 1945 Guelma

Faculty of Science and Technology

Algebra 2 – Home Work

"يُرجى تسليم جميع الإجابات في الحصة الأخيرة من هذا السداسي".

"يُرجى تقديم حلول مفصلة مع توضيح جميع خطوات الحساب،
وذلك باتباع نفس المنهجية المعتمدة في الأعمال الموجهة (TD)".

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Exercise 1:
Let $h : \mathbb{R}^4 \to \mathbb{R}^3$ be the map defined by

$$h\left(\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}\right) = \begin{pmatrix} x+y+z \\ x+ay+bz \\ x+y+z+t \end{pmatrix},$$

where $a, b \in \mathbb{R}$ are parameters.

(1) Prove that h is linear.

(2) Determine $\ker h$ and $\operatorname{Im} h$. Give a basis for each. (The answer may depend on a and b.)

(3) Compute Nullity(h) and Rank(h). For which values of a and b is h injective? Surjective? Justify.

(4) Determine whether h is an isomorphism. Justify.

(5) Repeat Questions 1–4, providing full computations, in the following cases, $a = b = 1$ and $a = -1, b = 6$.

Exercise 2:
Consider the following system of three linear equations in three unknowns, depending on parameters $\alpha, \beta \in \mathbb{R}$:

$$\begin{cases} x_1 - x_2 = \alpha, \\ x_1 - x_2 + x_3 = \beta, \\ x_1 + 2x_2 + x_3 = 1. \end{cases}$$

(1) Write the system in matrix form $Ax = b$, specifying A, x, and b.

(2) Compute $\det A$. Is the system uniquely solvable for all α, β? Justify.

(3) Use Cramer's rule to find x_1, x_2, x_3 in terms of α and β.

(4) Determine A^{-1} and verify that $(x_1, x_2, x_3)^T = A^{-1}b$.

(5) Repeat Questions 1–4, providing full computations, in the following cases, $\alpha = 1, \beta = -1$ and $\alpha = 3, \beta = -1$.

Exercise 3:
Let

$$A = \begin{pmatrix} p+q & -q & 0 \\ -q & p+2q & -q \\ 0 & -q & p+q \end{pmatrix}, \qquad p, q \in \mathbb{C},$$

and define

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

(1) Prove that $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{C}^3$.

(2) Show that u_1, u_2, u_3 are eigenvectors of A (i.e. $Au_i = \lambda_i u_i$, $i = 1, 2, 3$).

(3) Determine all eigenvalues of A (i.e. λ_i, $i = 1, 2, 3$).

(4) Deduce the characteristic polynomial of A (i.e. $P_A(\lambda) = \det(A - \lambda I_3)$).

(5) Compute $D = P^{-1}AP$ where $P = (u_1\ u_2\ u_3)$.

"Stay calm, think wisely, and succeed!"

Solution 1:
We analyse the properties of h in full generality, then apply the results to two specific pairs of parameters.

1. Linearity

Take arbitrary vectors

$$u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}, \qquad v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix} \in \mathbb{R}^4, \qquad \lambda \in \mathbb{R}.$$

Additivity,

$$u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ t_1 + t_2 \end{pmatrix},$$

$$\begin{aligned} h(u+v) &= \begin{pmatrix} (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \\ (x_1 + x_2) + a(y_1 + y_2) + b(z_1 + z_2) \\ (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) + (t_1 + t_2) \end{pmatrix} \\ &= \begin{pmatrix} (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) \\ (x_1 + ay_1 + bz_1) + (x_2 + ay_2 + bz_2) \\ (x_1 + y_1 + z_1 + t_1) + (x_2 + y_2 + z_2 + t_2) \end{pmatrix} \\ &= h(u) + h(v). \end{aligned}$$

Homogeneity,

$$\lambda u = \begin{pmatrix} \lambda x_1 \\ \lambda y_1 \\ \lambda z_1 \\ \lambda t_1 \end{pmatrix},$$

$$h(\lambda u) = \begin{pmatrix} \lambda x_1 + \lambda y_1 + \lambda z_1 \\ \lambda x_1 + a(\lambda y_1) + b(\lambda z_1) \\ \lambda x_1 + \lambda y_1 + \lambda z_1 + \lambda t_1 \end{pmatrix} = \lambda \begin{pmatrix} x_1 + y_1 + z_1 \\ x_1 + ay_1 + bz_1 \\ x_1 + y_1 + z_1 + t_1 \end{pmatrix} = \lambda\, h(u).$$

Hence h is a linear map for every $a, b \in \mathbb{R}$.

2. Kernel of h

$$\text{Ker}(h) = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; h\left(\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}\right) = \begin{pmatrix} x + y + z \\ x + ay + bz \\ x + y + z + t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

This yields the homogeneous system

$$\begin{cases} x + y + z = 0 & (1) \\ x + ay + bz = 0 & (2) \\ x + y + z + t = 0 & (3) \end{cases}$$

From (1) and (3) we immediately obtain $t = 0$. From (1) we have $x = -y - z$. Substituting into (2) gives

$$(-y - z) + ay + bz = 0 \implies (a - 1)y + (b - 1)z = 0. \tag{4}$$

We distinguish four cases based on the values of a and b.

- **Case 1,** $a = 1,\ b = 1$**.** Equation (4) is $0 = 0$; it imposes no restriction. Hence

$$\text{Ker}(h) = \left\{ \begin{pmatrix} -y - z \\ y \\ z \\ 0 \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Linear independence check

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that,

$$\lambda_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0_{W=\mathbb{R}^4}.$$

This yields,

$$\begin{cases} -\lambda_1 - \lambda_2 = 0 \\ \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases} \implies \lambda_1 = \lambda_2 = 0. \tag{0.1}$$

A basis is $\mathcal{B}_{\text{Ker}} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

- **Case 2,** $a = 1,\ b \neq 1$**.** Then (4) becomes $(b-1)z = 0$, and since $b - 1 \neq 0$ we get $z = 0$. Consequently $x = -y$ and $t = 0$.

$$\text{Ker}(h) = \left\{ \begin{pmatrix} -y \\ y \\ 0 \\ 0 \end{pmatrix} \middle| \ y \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Basis, $\mathcal{B}_{\text{Ker}} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

- **Case 3,** $a \neq 1,\ b = 1$**.** Now (4) gives $(a-1)y = 0$, whence $y = 0$. So $x = -z$ and $t = 0$.

$$\text{Ker}(h) = \left\{ \begin{pmatrix} -z \\ 0 \\ z \\ 0 \end{pmatrix} \middle| \ z \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Basis, $\mathcal{B}_{\text{Ker}} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

- **Case 4,** $a \neq 1,\ b \neq 1$**.** From (4) we solve for y,

$$y = -\frac{b-1}{a-1} z.$$

Then

$$x = -y - z = \frac{b-1}{a-1} z - z = \frac{b-a}{a-1} z.$$

Thus

$$\text{Ker}(h) = \left\{ z \begin{pmatrix} \frac{b-a}{a-1} \\ -\frac{b-1}{a-1} \\ 1 \\ 0 \end{pmatrix} \middle| \ z \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} b-a \\ -(b-1) \\ a-1 \\ 0 \end{pmatrix} \right\}.$$

Basis, $\mathcal{B}_{\text{Ker}} = \left\{ \begin{pmatrix} b-a \\ -(b-1) \\ a-1 \\ 0 \end{pmatrix} \right\}$.

Image of h

By definition,

$$\mathrm{Img}(h)=\left\{h\left(\begin{pmatrix}x\\y\\z\\t\end{pmatrix}\right)=\begin{pmatrix}x+y+z\\x+ay+bz\\x+y+z+t\end{pmatrix}=x\begin{pmatrix}1\\1\\1\end{pmatrix}+y\begin{pmatrix}1\\a\\1\end{pmatrix}+z\begin{pmatrix}1\\b\\1\end{pmatrix}+t\begin{pmatrix}0\\0\\1\end{pmatrix}\;\middle|\; x,y,z,t\in\mathbb{R}\right\}$$

$$=\mathrm{Span}\left\{w_1=\begin{pmatrix}1\\1\\1\end{pmatrix},\ w_2=\begin{pmatrix}1\\a\\1\end{pmatrix},\ w_3=\begin{pmatrix}1\\b\\1\end{pmatrix},\ w_4=\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

We determine a maximal linearly independent subset. Consider the equation

$$\lambda_1 w_1+\lambda_2 w_2+\lambda_3 w_3+\lambda_4 w_4=\mathbf{0}_{\mathbb{R}^3},$$

which componentwise reads

$$\begin{cases}\lambda_1+\lambda_2+\lambda_3=0 & (I)\\ \lambda_1+a\lambda_2+b\lambda_3=0 & (II)\\ \lambda_1+\lambda_2+\lambda_3+\lambda_4=0 & (III)\end{cases}$$

From (I) and (III) we get $\lambda_4=0$. Subtracting (I) from (II) yields

$$(a-1)\lambda_2+(b-1)\lambda_3=0. \tag{IV}$$

- **Subcase A, $(a,b)\neq(1,1)$.** We show that three of the vectors are always linearly independent, hence $\dim\mathrm{Img}(h)=3$ and $\mathrm{Img}(h)=\mathbb{R}^3$.
  - If $a\neq 1$, consider $\{w_1,w_2,w_4\}$. Setting $\lambda_1 w_1+\lambda_2 w_2+\lambda_4 w_4=0$ gives

    $$\begin{cases}\lambda_1+\lambda_2=0,\\ \lambda_1+a\lambda_2=0,\\ \lambda_1+\lambda_2+\lambda_4=0.\end{cases}$$

    From $\lambda_2=-\lambda_1$ and $(1-a)\lambda_1=0$ we obtain $\lambda_1=0$ (since $a\neq 1$), hence $\lambda_2=\lambda_4=0$.
  - If $a=1$ but $(a,b)\neq(1,1)$, then necessarily $b\neq 1$. The set $\{w_1,w_3,w_4\}$ similarly gives $\lambda_1=0$ and consequently all $\lambda_i=0$.

  Therefore $\mathrm{Img}(h)=\mathbb{R}^3$; any basis of $\mathbb{R}^3$ (for instance the standard basis) is a basis of the image.
- **Subcase B, $a=1$ and $b=1$.** Equation (IV) disappears and (I) simply reads $\lambda_1+\lambda_2+\lambda_3=0$. Thus $w_1=w_2=w_3$, and the only independent vectors are w_1 and w_4. Hence

  $$\mathrm{Img}(h)=\mathrm{Span}\{w_1,w_4\}=\mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

  Linear independence check

Let $\lambda_1,\lambda_2\in\mathbb{R}$ such that,

$$\lambda_1\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda_2\begin{pmatrix}0\\0\\1\end{pmatrix}=0_{W=\mathbb{R}^3}.$$

This yields,

$$\begin{cases}\lambda_1=0\\ \lambda_2=0\end{cases}\implies\lambda_1=\lambda_2=0. \tag{0.2}$$

A basis is $\mathcal{B}_{\mathrm{Img}}=\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

3. Nullity, rank, injectivity, surjectivity

From the kernel analysis,

$$\text{Nullity}(h) = \dim \text{Ker}(h) = \begin{cases} 2, & a = b = 1, \\ 1, & \text{otherwise.} \end{cases}$$

By the Rank–Nullity Theorem, $\dim \text{Ker}(h) + \dim \text{Img}(h) = \dim \mathbb{R}^4 = 4$, therefore

$$\text{Rank}(h) = \dim \text{Im}(h) = \begin{cases} 2, & a = b = 1, \\ 3, & \text{otherwise.} \end{cases}$$

- **Injectivity,** h is injective iff $\text{Ker}(h) = \{0\}$. Since $\dim \text{Ker}(h) \geq 1$ for all a, b, the map is never injective.
- **Surjectivity,** h is surjective iff $\text{Im}(h) = \mathbb{R}^3$ (i.e. $\text{Im}(h) \subseteq \mathbb{R}^3$ and $\text{rank}(h) = 3$). This occurs precisely when $(a, b) \neq (1, 1)$. For $a = b = 1$ the map is not surjective.

4. Isomorphism

A linear map is an isomorphism iff it is simultaneously injective and surjective. Because injectivity always fails, h is not an isomorphism for any choice of $a, b \in \mathbb{R}$.

5. Case, $a = 1,\ b = 1$ (direct verification)

For completeness we solve the case $a = b = 1$ independently and check consistency with the general results. 1. Linearity

Take arbitrary vectors

$$u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}, \qquad v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix} \in \mathbb{R}^4, \qquad \lambda \in \mathbb{R}.$$

Additivity,

$$u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ t_1 + t_2 \end{pmatrix},$$

$$\begin{aligned} h(u+v) &= \begin{pmatrix} (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \\ (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \\ (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) + (t_1 + t_2) \end{pmatrix} \\ &= \begin{pmatrix} (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) \\ (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) \\ (x_1 + y_1 + z_1 + t_1) + (x_2 + y_2 + z_2 + t_2) \end{pmatrix} \\ &= h(u) + h(v). \end{aligned}$$

Homogeneity,

$$\lambda u = \begin{pmatrix} \lambda x_1 \\ \lambda y_1 \\ \lambda z_1 \\ \lambda t_1 \end{pmatrix},$$

$$h(\lambda u) = \begin{pmatrix} \lambda x_1 + \lambda y_1 + \lambda z_1 \\ \lambda x_1 + (\lambda y_1) + (\lambda z_1) \\ \lambda x_1 + \lambda y_1 + \lambda z_1 + \lambda t_1 \end{pmatrix} = \lambda \begin{pmatrix} x_1 + y_1 + z_1 \\ x_1 + y_1 + z_1 \\ x_1 + y_1 + z_1 + t_1 \end{pmatrix} = \lambda\, h(u).$$

Hence h is a linear map.

2. Kernel of h

$$\mathrm{Ker}(h) = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; h\left(\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}\right) = \begin{pmatrix} x + y + z \\ x + y + z \\ x + y + z + t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

This yields the homogeneous system

$$\begin{cases} x + y + z = 0 & (1) \\ x + y + z = 0 & (2) \\ x + y + z + t = 0 & (3) \end{cases}$$

From (1) and (3) we immediately obtain $t = 0$. From (1) we have $x = -y - z$.

- Equation (2) is $0 = 0$; it imposes no restriction. Hence

$$\mathrm{Ker}(h) = \left\{ \begin{pmatrix} -y - z \\ y \\ z \\ 0 \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\} = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Linear independence check

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that,

$$\lambda_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0_{W=\mathbb{R}^4}.$$

This yields,

$$\begin{cases} -\lambda_1 - \lambda_2 = 0 \\ \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases} \implies \lambda_1 = \lambda_2 = 0. \tag{0.3}$$

A basis is $\mathcal{B}_{\text{Ker}} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Image of h

By definition,

$$\text{Img}(h) = \left\{ h\left(\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}\right) = \begin{pmatrix} x+y+z \\ x+y+z \\ x+y+z+t \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + z\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, x, y, z, t \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We determine a maximal linearly independent subset. Consider the equation

$$\lambda_1 w_1 + \lambda_2 w_2 = \mathbf{0}_{\mathbb{R}^3},$$

which componentwise reads

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$$

Hence

$$\text{Img}(h) = \text{Span}\{w_1, w_4\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Linear independence check

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that,

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0_{W=\mathbb{R}^3}.$$

This yields,

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases} \implies \lambda_1 = \lambda_2 = 0. \tag{0.4}$$

A basis is $\mathcal{B}_{\text{Img}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

3. Nullity, rank, injectivity, surjectivity

From the kernel analysis,

$$\text{Nullity}(h) = \dim \text{Ker}(h) = 2.$$

By the Rank–Nullity Theorem, $\dim \text{Ker}(h) + \dim \text{Img}(h) = \dim \mathbb{R}^4 = 4$, therefore

$$\text{Rank}(h) = \dim \text{Im}(h) = 2.$$

- **Injectivity,** h is injective iff $\text{Ker}(h) = \{0\}$. Since $\dim \text{Ker}(h) \geq 1$, the map is never injective.
- **Surjectivity,** h is surjective iff $\text{Im}(h) = \mathbb{R}^3$ (i.e. $\text{Im}(h) \subseteq \mathbb{R}^3$ and $\text{rank}(h) = 3$). For $\dim \text{Im}(h) = 2$ the map is not surjective.

4. Isomorphism

A linear map is an isomorphism iff it is simultaneously injective and surjective. Because injectivity always fails, h is not an isomorphism.

Case 2, $a = -1,\ b = 6$ (direct verification)

1. Linearity

Take arbitrary vectors

$$u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ t_1 \end{pmatrix}, \qquad v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ t_2 \end{pmatrix} \in \mathbb{R}^4, \qquad \lambda \in \mathbb{R}.$$

Additivity,

$$u + v = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \\ t_1 + t_2 \end{pmatrix},$$

$$\begin{aligned} h(u+v) &= \begin{pmatrix} (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \\ (x_1 + x_2) - (y_1 + y_2) + 6(z_1 + z_2) \\ (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) + (t_1 + t_2) \end{pmatrix} \\ &= \begin{pmatrix} (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) \\ (x_1 - y_1 + 6z_1) + (x_2 - y_2 + 6z_2) \\ (x_1 + y_1 + z_1 + t_1) + (x_2 + y_2 + z_2 + t_2) \end{pmatrix} \\ &= h(u) + h(v). \end{aligned}$$

Homogeneity,

$$\lambda u = \begin{pmatrix} \lambda x_1 \\ \lambda y_1 \\ \lambda z_1 \\ \lambda t_1 \end{pmatrix},$$

$$h(\lambda u) = \begin{pmatrix} \lambda x_1 + \lambda y_1 + \lambda z_1 \\ \lambda x_1 - (\lambda y_1) + 6(\lambda z_1) \\ \lambda x_1 + \lambda y_1 + \lambda z_1 + \lambda t_1 \end{pmatrix} = \lambda \begin{pmatrix} x_1 + y_1 + z_1 \\ x_1 - y_1 + 6z_1 \\ x_1 + y_1 + z_1 + t_1 \end{pmatrix} = \lambda\, h(u).$$

Hence h is a linear map.

2. Kernel of h

$$\operatorname{Ker}(h) = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; h\left(\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}\right) = \begin{pmatrix} x + y + z \\ x - y + 6z \\ x + y + z + t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

This yields the homogeneous system

$$\begin{cases} x + y + z = 0 & (1) \\ x - y + 6z = 0 & (2) \\ x + y + z + t = 0 & (3) \end{cases}$$

From (1) and (3) we immediately obtain $t = 0$. From (1) we have $x = -y - z$. Substituting into (2) gives

$$(-y - z) - y + 6z = 0 \implies -2y + 5z = 0. \tag{4}$$

- From (4) we solve for y,

  $$y = -\frac{5}{-2}\, z.$$

  Then

  $$x = -y - z = \frac{5}{-2} z - z = \frac{7}{-2}\, z.$$

Thus

$$\operatorname{Ker}(h) = \left\{ z \begin{pmatrix} -\frac{7}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{pmatrix} \middle| \; z \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} 7 \\ 5 \\ -2 \\ 0 \end{pmatrix} \right\}.$$

Basis, $\mathcal{B}_{\text{Ker}} = \left\{ \begin{pmatrix} 7 \\ 5 \\ -2 \\ 0 \end{pmatrix} \right\}.$

Image of h

By definition,

$$\operatorname{Img}(h) = \left\{ h\left(\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \right) = \begin{pmatrix} x+y+z \\ x-y+6z \\ x+y+z+t \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + z\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \; x, y, z, t \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left\{ w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; w_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \; w_3 = \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}, \; w_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We determine a maximal linearly independent subset. Consider the equation

$$\lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 + \lambda_4 w_4 = \mathbf{0}_{\mathbb{R}^3},$$

which componentwise reads

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 & (I) \\ \lambda_1 - \lambda_2 + 6\lambda_3 = 0 & (II) \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 & (III) \end{cases}$$

From (I) and (III) we get $\lambda_4 = 0$. Subtracting (I) from (II) yields

$$-2\lambda_2 + 5\lambda_3 = 0. \tag{IV}$$

- We show that three of the vectors are always linearly independent, hence $\dim \operatorname{Img}(h) = 3$ and $\operatorname{Img}(h) = \mathbb{R}^3$.
  - Consider $\{w_1, w_2, w_4\}$. Setting $\lambda_1 w_1 + \lambda_2 w_2 + \lambda_4 w_4 = 0$ gives

    $$\begin{cases} \lambda_1 + \lambda_2 = 0, \\ \lambda_1 - \lambda_2 = 0, \\ \lambda_1 + \lambda_2 + \lambda_4 = 0. \end{cases}$$

    From $\lambda_2 = -\lambda_1$ and $2\lambda_1 = 0$ we obtain $\lambda_1 = 0$, hence $\lambda_2 = \lambda_4 = 0$.

  Therefore $\operatorname{Img}(h) = \mathbb{R}^3$; any basis of $\mathbb{R}^3$ (for instance the standard basis) is a basis of the image.

3. Nullity, rank, injectivity, surjectivity

From the kernel analysis,

$$\operatorname{Nullity}(h) = \dim \operatorname{Ker}(h) = 1.$$

By the Rank–Nullity Theorem, $\dim \operatorname{Ker}(h) + \dim \operatorname{Img}(h) = \dim \mathbb{R}^4 = 4$, therefore

$$\operatorname{Rank}(h) = \dim \operatorname{Im}(h) = 3$$

- **Injectivity,** h is injective iff $\operatorname{Ker}(h) = \{0\}$. Since $\dim \operatorname{Ker}(h) \geq 1$, the map is never injective.
- **Surjectivity,** h is surjective iff $\operatorname{Im}(h) = \mathbb{R}^3$ (i.e. $\operatorname{Im}(h) \subseteq \mathbb{R}^3$ and $\operatorname{rank}(h) = 3$). We have $\operatorname{Im}(h) \subseteq \mathbb{R}^3$ and $\dim \operatorname{Im}(h) = 3 \Longrightarrow$ The map is surjective.

4. Isomorphism

The map is surjective but not injective, therefore not an isomorphism.

Solution 2:

1. Matrix form

Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad b = \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}.$$

The system is $Ax = b$.

2. Determinant of A

Expanding along the first row,

$$\det A = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 0.$$

$$\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = (-1) \cdot 1 - 1 \cdot 2 = -3, \qquad \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

Hence $\det A = -3 \neq 0$. The matrix is invertible and the system has a unique solution for every α, β.

3. Cramer's rule

Let A_j be the matrix obtained by replacing column j of A with b.

Computation of x_1,

$$A_1 = \begin{pmatrix} \alpha & -1 & 0 \\ \beta & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Expanding along the first row,

$$\det A_1 = \alpha \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - (-1) \begin{vmatrix} \beta & 1 \\ 1 & 1 \end{vmatrix} = \alpha(-3) + (\beta - 1) = -3\alpha + \beta - 1.$$

Thus

$$x_1 = \frac{\det A_1}{\det A} = \frac{-3\alpha + \beta - 1}{-3} = \frac{3\alpha - \beta + 1}{3}.$$

Computation of x_2,

$$A_2 = \begin{pmatrix} 1 & \alpha & 0 \\ 1 & \beta & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

$$\det A_2 = 1 \cdot \begin{vmatrix} \beta & 1 \\ 1 & 1 \end{vmatrix} - \alpha \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = (\beta - 1) - \alpha \cdot 0 = \beta - 1.$$

$$x_2 = \frac{\beta - 1}{-3} = \frac{1 - \beta}{3}.$$

Computation of x_3,

$$A_3 = \begin{pmatrix} 1 & -1 & \alpha \\ 1 & -1 & \beta \\ 1 & 2 & 1 \end{pmatrix}.$$

Expanding along the first row (minor expansions),

$$\begin{aligned} \det A_3 &= 1 \cdot \begin{vmatrix} -1 & \beta \\ 2 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & \beta \\ 1 & 1 \end{vmatrix} + \alpha \cdot \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} \\ &= (-1 - 2\beta) + (1 - \beta) + \alpha\,(2 + 1) \\ &= -2\beta - \beta + 3\alpha = 3\alpha - 3\beta. \end{aligned}$$

$$x_3 = \frac{3\alpha - 3\beta}{-3} = \beta - \alpha.$$

Hence the unique solution is

$$\boxed{x_1 = \frac{3\alpha - \beta + 1}{3}, \qquad x_2 = \frac{1-\beta}{3}, \qquad x_3 = \beta - \alpha.}$$

4. Inverse of A and verification

Using the adjugate formula $A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A)$. First compute the cofactors,

$$\begin{aligned}
C_{11} &= +\begin{vmatrix}-1 & 1\\ 2 & 1\end{vmatrix} = -3, & C_{12} &= -\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix} = 0, & C_{13} &= +\begin{vmatrix}1 & -1\\ 1 & 2\end{vmatrix} = 3,\\
C_{21} &= -\begin{vmatrix}-1 & 0\\ 2 & 1\end{vmatrix} = 1, & C_{22} &= +\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix} = 1, & C_{23} &= -\begin{vmatrix}1 & -1\\ 1 & 2\end{vmatrix} = -3,\\
C_{31} &= +\begin{vmatrix}-1 & 0\\ -1 & 1\end{vmatrix} = -1, & C_{32} &= -\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix} = -1, & C_{33} &= +\begin{vmatrix}1 & -1\\ 1 & -1\end{vmatrix} = 0.
\end{aligned}$$

The cofactor matrix is $\begin{pmatrix}-3 & 0 & 3\\ 1 & 1 & -3\\ -1 & -1 & 0\end{pmatrix}$. Its transpose gives the adjugate,

$$\mathrm{adj}(A) = \begin{pmatrix}-3 & 1 & -1\\ 0 & 1 & -1\\ 3 & -3 & 0\end{pmatrix}.$$

Since $\det A = -3$,

$$A^{-1} = \frac{1}{-3}\begin{pmatrix}-3 & 1 & -1\\ 0 & 1 & -1\\ 3 & -3 & 0\end{pmatrix} = \begin{pmatrix}1 & -\frac{1}{3} & \frac{1}{3}\\ 0 & -\frac{1}{3} & \frac{1}{3}\\ -1 & 1 & 0\end{pmatrix}.$$

Direct multiplication confirms $x = A^{-1}b$ yields the same solution as Cramer's rule.

5. Special numeric cases

- $\alpha = 1,\ \beta = -1, x_1 = \frac{3\cdot 1-(-1)+1}{3} = \frac{5}{3},\ x_2 = \frac{1-(-1)}{3} = \frac{2}{3},\ x_3 = -1-1 = -2.$
- $\alpha = 3,\ \beta = -1, x_1 = \frac{9+1+1}{3} = \frac{11}{3},\ x_2 = \frac{2}{3},\ x_3 = -1-3 = -4.$

Solution 3:

1. The set $\{u_1, u_2, u_3\}$ is a basis of $\mathbb{C}^3$

Form the matrix

$$P = \begin{pmatrix} u_1 \, u_2 \, u_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}.$$

Its determinant (expanding along the first row),

$$\begin{aligned} \det P &= 1 \cdot \begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ &= (0 \cdot 1 - (-2) \cdot (-1)) \; - \; (1 \cdot 1 - (-2) \cdot 1) \; + \; (1 \cdot (-1) - 0 \cdot 1) \\ &= -2 \; - \; 3 \; - \; 1 = -6 \neq 0. \end{aligned}$$

Since $\det P \neq 0$, the columns are linearly independent. In a threedimensional space three independent vectors automatically form a basis.

Method 2

Linear independence check

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that,

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This yields,

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 0 & \text{(a)} \\ \lambda_1 - 2\lambda_3 = 0 & \text{(b)} \\ \lambda_1 - \lambda_2 + \lambda_3 = 0 & \text{(c)} \end{cases} \tag{0.5}$$

1. From (b), $\lambda_1 = 2\lambda_3$;
2. Substitute into (a), $2\lambda_3 + \lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_2 = -3\lambda_3$;
3. Substitute into (c), $2\lambda_3 + 3\lambda_3 + \lambda_3 = 0 \Rightarrow 6\lambda_3 = 0 \Rightarrow \lambda_3 = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 0.$

Thus $\lambda_1 = \lambda_2 = \lambda_3 = 0$, so the vectors are linearly independent. In a threedimensional space three independent vectors automatically form a basis.

2. Verification that u_1, u_2, u_3 are eigenvectors of A

Compute the products,

$$\begin{aligned} Au_1 &= \begin{pmatrix} p+q & -q & 0 \\ -q & p+2q & -q \\ 0 & -q & p+q \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} (p+q) - q \\ -q + (p+2q) - q \\ -q + (p+q) \end{pmatrix} = \begin{pmatrix} p \\ p \\ p \end{pmatrix} = p \, u_1. \end{aligned}$$

$$\begin{aligned} Au_2 &= \begin{pmatrix} p+q & -q & 0 \\ -q & p+2q & -q \\ 0 & -q & p+q \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} (p+q) + 0 \\ -q + 0 + q \\ 0 + 0 - (p+q) \end{pmatrix} = \begin{pmatrix} p+q \\ 0 \\ -p-q \end{pmatrix} = (p+q) \, u_2. \end{aligned}$$

$$\begin{aligned} Au_3 &= \begin{pmatrix} p+q & -q & 0 \\ -q & p+2q & -q \\ 0 & -q & p+q \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} (p+q) + 2q \\ -q - 2(p+2q) - q \\ 2q + (p+q) \end{pmatrix} = \begin{pmatrix} p+3q \\ -2p-6q \\ p+3q \end{pmatrix} = (p+3q) \, u_3. \end{aligned}$$

Thus u_1, u_2, u_3 are eigenvectors of A.

3. Eigenvalues

From the above, the eigenvalues of A are

$$\boxed{\lambda_1 = p, \quad \lambda_2 = p + q, \quad \lambda_3 = p + 3q.}$$

4. Characteristic polynomial

Because we have found three linearly independent eigenvectors, A is diagonalisable and its characteristic polynomial is the product of the factors $(\lambda - \lambda_i)$,

$$\begin{aligned} P_A(\lambda) &= (-1)^{n=3}(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \\ & -(\lambda - p)\big(\lambda - (p+q)\big)\big(\lambda - (p+3q)\big). \end{aligned}$$

5. Diagonalisation $D = P^{-1}AP$

Let P be the matrix whose columns are u_1, u_2, u_3. From the eigenvector relations,

$$AP = A(u_1\ u_2\ u_3) = (Au_1\ Au_2\ Au_3) = (\lambda_1 u_1\ \lambda_2 u_2\ \lambda_3 u_3) = P\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Multiplying on the left by P^{-1} (which exists because $\det P \neq 0$),

$$D = P^{-1}AP = \begin{pmatrix} p & 0 & 0 \\ 0 & p+q & 0 \\ 0 & 0 & p+3q \end{pmatrix}.$$