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1. Solutions of Tutorial Exercises (1):

A. LESLOUS A. AYACHI

1.1. Preliminary

Let V be a set equipped with two operations,

- Addition: $+ : V \times V \to V$, $(\mathrm{u}, \mathrm{v}) \mapsto \mathrm{u} + \mathrm{v}$
- Scalar multiplication: $\cdot : \mathbb{F} \times V \to V$, $(\lambda, \mathrm{v}) \mapsto \lambda \mathrm{v}$ (where $\mathbb{F} = (\mathbb{R}$ or $\mathbb{C}))$

such that all ten vector space axioms are satisfied.

For $n \in \mathbb{N}^*$, an element of $V = \mathbb{F}^n$ is written as a column vector,

$$\mathrm{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{pmatrix} \in \mathbb{F}^n$$

or equivalently as $\mathrm{v}_1 = (v_{11}, v_{21}, \dots, v_{n1})^T$.

For $\mathrm{v}_i = (v_{1i}, \dots, v_{ni})^T \in V = \mathbb{F}^n$, and $\lambda_i \in \mathbb{F}$ for $i = 1, \cdots, k$,

$$\mathrm{v}_1 + \mathrm{v}_2 = \begin{pmatrix} v_{11} + v_{12} \\ \vdots \\ v_{n1} + v_{n2} \end{pmatrix} \qquad \lambda_1 \mathrm{v}_1 = \begin{pmatrix} \lambda_1 v_{11} \\ \vdots \\ \lambda_1 v_{n1} \end{pmatrix}. \tag{1.1}$$

$$\begin{aligned} \sum_{i=1}^{k} \lambda_i \mathrm{v}_i &= \lambda_1 \mathrm{v}_1 + \lambda_2 \mathrm{v}_2 + \cdots + \lambda_k \mathrm{v}_\mathrm{k} \\ &= \lambda_1 \begin{pmatrix} v_{11} \\ \vdots \\ v_{n1} \end{pmatrix} + \lambda_2 \begin{pmatrix} v_{12} \\ \vdots \\ v_{n2} \end{pmatrix} + \cdots + \lambda_k \begin{pmatrix} v_{1k} \\ \vdots \\ v_{nk} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \lambda_1 v_{11} + \lambda_2 v_{12} + \cdots + \lambda_k v_{1k} \\ \vdots \\ \lambda_1 v_{n1} + \lambda_2 v_{n2} + \cdots + \lambda_k v_{nk} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{k} \lambda_i v_{1i} \\ \vdots \\ \sum_{i=1}^{k} \lambda_i v_{ni} \end{pmatrix}. \tag{1.2}$$

Definition 1.1 (Linear combination):
Given vectors $\mathrm{v}_1, \ldots, \mathrm{v}_k \in V$ and scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$, the vector $\lambda_1 \mathrm{v}_1 + \cdots + \lambda_k \mathrm{v}_k$ is called a linear combination.

Definition 1.2 (Linear independence):
Vectors $\{\mathrm{v}_1, \ldots, \mathrm{v}_k\}$ are linearly independent if the equation

$$\lambda_1 \mathrm{v}_1 + \cdots + \lambda_k \mathrm{v}_k = 0$$

implies $\lambda_1 = \cdots = \lambda_k = 0$.

Definition 1.3 (Span):
The span of $\{\mathrm{v}_1, \ldots, \mathrm{v}_k\}$ is,

$$\mathrm{Span}\{\mathrm{v}_1, \ldots, \mathrm{v}_k\} = \{\lambda_1 \mathrm{v}_1 + \cdots + \lambda_k \mathrm{v}_k : \lambda_1, \ldots, \lambda_k \in \mathbb{F}\}.$$

Definition 1.4 (Linear map):
A function $f : V \to W$ between vector spaces is linear if,

$$f(\mathrm{u} + \mathrm{v}) = f(\mathrm{u}) + f(\mathrm{v}) \quad \forall \mathrm{u}, \mathrm{v} \in V, \tag{1.3}$$
$$f(\lambda \mathrm{v}) = \lambda f(\mathrm{v}) \quad \forall \lambda \in \mathbb{F}, \forall \mathrm{v} \in V. \tag{1.4}$$

Or

$$f(\lambda_1 \mathrm{v}_1 + \lambda_2 \mathrm{v}_2) = \lambda_1 f(\mathrm{v}_1) + \lambda_2 f(\mathrm{v}_2), \quad \forall \lambda_1, \lambda_2 \in \mathbb{F}, \forall \mathrm{v}_1, \mathrm{v}_2 \in V.$$

Definition 1.5 (Kernel and Image):
For a linear map $f : V \to W$,

$$\mathrm{Ker}(f) = \{\mathrm{v} \in V : f(\mathrm{v}) = 0_W\} \subseteq V, \tag{1.5}$$
$$\mathrm{Im}(f) = \{f(\mathrm{v}) : \mathrm{v} \in V\} \subseteq W. \tag{1.6}$$

Definition 1.6 (Basis and dimension):
A set $\{\mathrm{v}_1, \ldots, \mathrm{v}_k\}$ is a basis of V if it is linearly independent and spans V. The dimension $\dim(V)$ is the number of vectors in a basis.

Definition 1.7 (Direct sum):
$V = U \oplus W$ if every $\mathrm{v} \in V$ can be written uniquely as $\mathrm{v} = \mathrm{u} + \mathrm{w}$ with $\mathrm{u} \in U$, $\mathrm{w} \in W$. Equivalently, $U \cap W = \{0_V\}$ and $U + W = V$.

Lemma 1.1 (Kernel characterization of injectivity):
A linear map $f : V \to W$ is,

- Injective $\iff \mathrm{Ker}(f) = \{0_V\}$.
- Surjective $\iff \mathrm{Im}(f) = W$.

- Bijective $\iff$ Injective and Surjective.
- Isomorphism $\iff$ Bijective and linear map.

Theorem 1.1 (Rank-Nullity Theorem):
For a linear map $f : V \to W$ with $\dim(V) < \infty$,

$$\underbrace{\text{Nullity}(f)}_{\dim(\text{Ker}(f))} + \underbrace{\text{Rank}(f)}_{\dim(\text{Im}(f))} = \dim(V).$$

Criterion 1.1 (Direct sum criterion):
For subspaces $U, W \subseteq V$,

$$V = U \oplus W \iff \begin{cases} U \cap W = \{0_V\}, \\ \dim(U) + \dim(W) = \dim(V). \end{cases}$$

Exercise 1.1:
Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be the map defined by,

$$f\left(\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}\right) = \begin{pmatrix} x-2y \\ x-2y-3z \\ 0 \\ x-y-z-t \end{pmatrix}.$$

1. Prove that f is is a linear map.
2. Determine $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$, giving a basis for each.
3. Decide whether f is injective, surjective, bijective, or an isomorphism.

Solution:

1. Proof of Linearity

   We verify the conditions of Definition 1.4. Let $\mathrm{u} = (x, y, z, t)^T$, $\mathrm{v} = (x', y', z', t')^T \in \mathbb{R}^4$, and $\lambda \in \mathbb{R}$.

   Additivity (1.3),

$$\begin{aligned} f(\mathrm{u}+\mathrm{v}) &= f\left(\begin{pmatrix} x+x' \\ y+y' \\ z+z' \\ t+t' \end{pmatrix}\right) \\ &= \begin{pmatrix} (x+x')-2(y+y') \\ (x+x')-2(y+y')-3(z+z') \\ 0 \\ (x+x')-(y+y')-(z+z')-(t+t') \end{pmatrix} \\ &= \begin{pmatrix} x-2y \\ x-2y-3z \\ 0 \\ x-y-z-t \end{pmatrix} + \begin{pmatrix} x'-2y' \\ x'-2y'-3z' \\ 0 \\ x'-y'-z'-t' \end{pmatrix} \\ &= f(\mathrm{u}) + f(\mathrm{v}). \end{aligned}$$

   Homogeneity (1.4),

$$f(\lambda \mathrm{u}) = f\left(\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda t \end{pmatrix}\right)$$

$$
\begin{aligned}
&= \begin{pmatrix} \lambda x - 2\lambda y \\ \lambda x - 2\lambda y - 3\lambda z \\ 0 \\ \lambda x - \lambda y - \lambda z - \lambda t \end{pmatrix} \\
&= \lambda \begin{pmatrix} x - 2y \\ x - 2y - 3z \\ 0 \\ x - y - z - t \end{pmatrix} \\
&= \lambda f(\mathrm{u}).
\end{aligned}
$$

Thus f satisfies both conditions of Definition 1.4 and is linear.

2. Determination of $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ with their basis

   a) Determination of $\mathrm{Ker}(f)$

   By Definition (1.5),

$$
\begin{aligned}
\mathrm{Ker}(f) &= \left\{ \mathrm{v} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, f(\mathrm{v}) = \begin{pmatrix} x - 2y \\ x - 2y - 3z \\ 0 \\ x - y - z - t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0_{W=\mathbb{R}^4} \right\} \\
&= \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \,\middle|\, \begin{cases} x - 2y = 0 & \text{(i)} \\ x - 2y - 3z = 0 & \text{(ii)} \\ 0 = 0 & \text{(iii)} \\ x - y - z - t = 0 & \text{(iv)} \end{cases} \right\}.
\end{aligned}
$$

   i. From (i), $x = 2y$;

   ii. Substitute into (ii), $2y - 2y - 3z = 0 \Rightarrow -3z = 0 \Rightarrow z = 0$;

   iii. Substitute into (iv), $2y - y - 0 - t = 0 \Rightarrow y - t = 0 \Rightarrow t = y$.

   Thus,

$$
\mathrm{v} = \begin{pmatrix} 2y \\ y \\ 0 \\ y \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \forall y \in \mathbb{R}.
$$

   By Definition 1.3,

$$
\mathrm{Ker}(f) = \mathrm{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.
$$

Since

$$\lambda \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \lambda = 0.$$

Basis of $\mathrm{Ker}(f)$, $\mathcal{B}_{\mathrm{Ker}} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Dimension, $\underbrace{\mathrm{Nullity}(f)}_{\dim(\mathrm{Ker}(f))} = 1$.

b) Determination of $\mathrm{Im}(f)$

$$\begin{aligned} \mathrm{Im}(f) &= \left\{ f(\mathrm{v}) = \begin{pmatrix} x-2y \\ x-2y-3z \\ 0 \\ x-y-z-t \end{pmatrix} \middle| \mathrm{v} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \right\} \\ &= \left\{ x \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} -2 \\ -2 \\ 0 \\ -1 \end{pmatrix} + z \begin{pmatrix} 0 \\ -3 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \right\} \end{aligned}$$

By Definition (1.6), $\mathrm{Im}(f)$ is spanned by the vectors in the linear combination,

$$\mathrm{Im}(f) = \mathrm{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Linear independence check

Apply Definition 1.2 to the first three vectors. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that,

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ -2 \\ 0 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ -3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0_{W=\mathbb{R}^4}.$$

This yields,

$$\begin{cases} \lambda_1 - 2\lambda_2 = 0 & \text{(a)} \\ \lambda_1 - 2\lambda_2 - 3\lambda_3 = 0 & \text{(b)} \\ \lambda_1 - \lambda_2 - \lambda_3 = 0 & \text{(c)} \end{cases} \tag{1.7}$$

i. From (a), $\lambda_1 = 2\lambda_2$;

ii. Substitute into (b), $2\lambda_2 - 2\lambda_2 - 3\lambda_3 = 0 \Rightarrow \lambda_3 = 0$;

iii. Substitute into (c), $2\lambda_2 - \lambda_2 - 0 = 0 \Rightarrow \lambda_2 = 0 \Rightarrow \lambda_1 = 0$.

Thus $\lambda_1 = \lambda_2 = \lambda_3 = 0$, so the vectors are linearly independent.

Basis of $\mathrm{Im}(f)$,

$$\mathcal{B}_{\mathrm{Im}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

3. Structural Properties: By Lemma 1.1,

- **Injectivity:** f is injective $\iff \mathrm{Ker}(f) = \{0_{\mathbb{R}^4}\}$. Since $\underbrace{\mathrm{Nullity}(f)}_{\dim(\mathrm{Ker}(f))} = 1 \neq 0$, f is NOT injective.

- **Surjectivity:** f is surjective $\iff \mathrm{Im}(f) = \mathbb{R}^4$. Since $\underbrace{\mathrm{Rank}(f)}_{\dim(\mathrm{Im}(f))} = 3 < 4 = \dim(\mathbb{R}^4)$, f is NOT surjective.

- **Bijectivity:** Since f is neither injective nor surjective, it is NOT bijective.

- **Isomorphism:** Since f is not bijective, it is NOT an isomorphism.

- Rank-Nullity Verification

  - Apply Theorem 1.1,
$$\underbrace{\mathrm{Nullity}(f)}_{\dim(\mathrm{Ker}(f))} + \underbrace{\mathrm{Rank}(f)}_{\dim(\mathrm{Im}(f))} = 1 + 3 = 4 = \dim(\mathbb{R}^4).$$
    This confirms the Rank-Nullity Theorem for f.

□

Exercise 1.2:
Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the map defined by,

$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x - z \\ 2x + y - 3z \\ -y + 2z \end{pmatrix}.$$

1. Prove that f is a linear map.

2. Determine the kernel $\text{Ker}(f)$ and the image $\text{Im}(f)$, and give a basis for each.

3. Compute the rank and check if f is bijective.

Solution:

1. Proof of Linearity

   We verify the conditions of Definition 1.4. Let $\mathrm{u} = (x, y, z)^T, \mathrm{v} = (x', y', z')^T \in \mathbb{R}^3$, and $\lambda \in \mathbb{R}$.

   Additivity,

$$\begin{aligned} f(\mathrm{u}+\mathrm{v}) &= f\left(\begin{pmatrix} x+x' \\ y+y' \\ z+z' \end{pmatrix}\right) \\ &= \begin{pmatrix} (x+x')-(z+z') \\ 2(x+x')+(y+y')-3(z+z') \\ -(y+y')+2(z+z') \end{pmatrix} \\ &= \begin{pmatrix} x-z \\ 2x+y-3z \\ -y+2z \end{pmatrix} + \begin{pmatrix} x'-z' \\ 2x'+y'-3z' \\ -y'+2z' \end{pmatrix} \\ &= f(\mathrm{u}) + f(\mathrm{v}). \end{aligned}$$

   Homogeneity,

$$\begin{aligned} f(\lambda\mathrm{u}) &= f\left(\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}\right) \\ &= \begin{pmatrix} \lambda x-\lambda z \\ 2\lambda x+\lambda y-3\lambda z \\ -\lambda y+2\lambda z \end{pmatrix} \\ &= \lambda \begin{pmatrix} x-z \\ 2x+y-3z \\ -y+2z \end{pmatrix} \\ &= \lambda f(\mathrm{u}). \end{aligned}$$

   Thus f is linear map.

2. Determination of $\text{Ker}(f)$ and $\text{Im}(f)$ with their basis

   a) Determination of $\text{Ker}(f)$

By Definition 1.5,

$$
\begin{aligned}
\mathrm{Ker}(f) &= \left\{ \mathrm{v} \in \mathbb{R}^3 : f(\mathrm{v}) = \begin{pmatrix} x - z \\ 2x + y - 3z \\ -y + 2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0_{W=\mathbb{R}^3} \right\} \\
&= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \begin{cases} x - z = 0 & (1) \\ 2x + y - 3z = 0 & (2) \\ -y + 2z = 0 & (3) \end{cases} \right\}.
\end{aligned}
$$

i. From equation (1), $x = z$;

ii. From equation (3), $y = 2z$;

iii. Substitute into equation (2), $2z + 2z - 3z = z = 0 \Rightarrow z = 0$;

iv. Thus $x = 0$, $y = 0$.

Therefore,

$$
\mathrm{Ker}(f) = \{0_{V=\mathbb{R}^3}\}.
$$

Since

$$
\lambda \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \not\Rightarrow \lambda = 0
$$

Basis, $\mathcal{B}_{\mathrm{Ker}} = \emptyset$ (the trivial subspace has no basis vectors).

b) Determination of $\mathrm{Im}(f)$

For $\mathrm{v} = (x, y, z)^T \in \mathbb{R}^3$,

$$
\begin{aligned}
\mathrm{Im}(f) &= \left\{ f(\mathrm{v}) = \begin{pmatrix} x - z \\ 2x + y - 3z \\ -y + 2z \end{pmatrix} \middle| \mathrm{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \right\} \\
&= \left\{ x \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + z \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \right\}
\end{aligned}
$$

Thus,

$$
\mathrm{Im}(f) = \mathrm{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} \right\}.
$$

Linear independence check

Apply Definition 1.2. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that,

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0_{W=\mathbb{R}^3}.$$

This gives,

$$\begin{cases} \lambda_1 - \lambda_3 = 0 & (1) \\ 2\lambda_1 + \lambda_2 - 3\lambda_3 = 0 & (2) \\ -\lambda_2 + 2\lambda_3 = 0 & (3) \end{cases}$$

i. From equation (1), $\lambda_1 = \lambda_3$;

ii. From equation (3), $\lambda_2 = 2\lambda_3$;

iii. Substitute into equation (2), $2\lambda_3 + 2\lambda_3 - 3\lambda_3 = \lambda_3 = 0$;

iv. Thus $\lambda_3 = 0$, then $\lambda_1 = 0$, $\lambda_2 = 0$.

So the vectors are linearly independent. Thus,

$$\mathcal{B}_{\mathrm{Im}} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} \right\} \text{ (or any basis of } W = \mathbb{R}^3\text{)}.$$

3. Rank and Bijectivity

Rank, $\underbrace{\mathrm{Rank}(f)}_{\dim(\mathrm{Im}(f))} = 3.$

Bijectivity check, By Lemma 1.1,

- **Injectivity:** Since $\mathrm{Ker}(f) = \{0_{\mathbb{R}^3}\}$, f is injective.
- **Surjectivity:** Since $\underbrace{\mathrm{Rank}(f)}_{\dim(\mathrm{Im}(f))} = 3$, $\mathrm{Im}(f) \subseteq \mathbb{R}^3 \Longleftrightarrow \mathrm{Im}(f) = \mathbb{R}^3$, f is surjective.

  since
- **Bijectivity:** Since f is both injective and surjective, it is bijective.

(Isomorphism: Since f is a bijective linear map, it is an isomorphism.)

- Rank-Nullity Verification

Apply Theorem 1.1,

$$\underbrace{\text{Nullity}(f)}_{\dim(\text{Ker}(f))} + \underbrace{\text{Rank}(f)}_{\dim(\text{Im}(f))} = 0 + 3 = 3 = \dim(\mathbb{R}^3).$$

$\square$

Exercise 1.3:
Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the map defined by,

$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y+z \\ x+y+z \\ x \end{pmatrix}$$

1. Prove that f is a linear map.
2. Determine the kernel $\text{Ker}(f)$ and the image $\text{Im}(f)$, and give a basis for each.
3. Decide whether f is an isomorphism and compute its rank.
4. Prove that $\mathbb{R}^3 = \text{Ker}(f) \oplus \text{Im}(f)$.

Solution:

1. Proof of Linearity

   We verify the conditions of Definition 1.4. Let $\text{u} = (x, y, z)^T, \text{v} = (x', y', z')^T \in \mathbb{R}^3$, and $\lambda \in \mathbb{R}$.

   Additivity,

$$\begin{aligned} f(\text{u}+\text{v}) &= f\left(\begin{pmatrix} x+x' \\ y+y' \\ z+z' \end{pmatrix}\right) \\ &= \begin{pmatrix} (y+y')+(z+z') \\ (x+x')+(y+y')+(z+z') \\ x+x' \end{pmatrix} \\ &= \begin{pmatrix} y+z \\ x+y+z \\ x \end{pmatrix} + \begin{pmatrix} y'+z' \\ x'+y'+z' \\ x' \end{pmatrix} \\ &= f(\text{u}) + f(\text{v}). \end{aligned}$$

Homogeneity,

$$\begin{aligned} f(\lambda \mathrm{u}) &= f\left(\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}\right) \\ &= \begin{pmatrix} \lambda y + \lambda z \\ \lambda x + \lambda y + \lambda z \\ \lambda x \end{pmatrix} \\ &= \lambda \begin{pmatrix} y + z \\ x + y + z \\ x \end{pmatrix} \\ &= \lambda f(\mathrm{u}). \end{aligned}$$

Thus f is linear map.

2. Determination of $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ with their basis

   a) Determination of $\mathrm{Ker}(f)$

      By Definition (1.5),

$$\begin{aligned} \mathrm{Ker}(f) &= \left\{ \mathrm{v} \in \mathbb{R}^3 \,\middle|\, f(\mathrm{v}) = \begin{pmatrix} y + z \\ x + y + z \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0_{W=\mathbb{R}^3} \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \begin{cases} y + z = 0 & (1) \\ x + y + z = 0 = 0 & (2) \\ x = 0 = 0 & (3) \end{cases} \right\}. \end{aligned}$$

      i. From equation (3), $x = 0$;

      ii. From equation (1), $z = -y$;

      iii. Check equation (2), $0 + y + (-y) = 0$ (satisfied).

      Thus,

$$\mathrm{v} = \begin{pmatrix} 0 \\ y \\ -y \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \forall y \in \mathbb{R}.$$

      Therefore,

$$\mathrm{Ker}(f) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Since

$$\lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \lambda = 0.$$

Thus,

$$\mathcal{B}_{\text{Ker}} = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

b) Determination of $\text{Im}(f)$

$$\begin{aligned} \text{Im}(f) &= \left\{ f(\text{v}) = \begin{pmatrix} y+z \\ x+y+z \\ x \end{pmatrix} \middle| \text{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \right\} \\ &= \left\{ x \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \right\} \end{aligned}$$

Thus,

$$\text{Im}(f) = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Linear independence check

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that,

$$\lambda_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0_{W=\mathbb{R}^3}.$$

This gives,

$$\begin{cases} \lambda_2 = 0 \\ \lambda_1 + \lambda_2 = 0 \\ \lambda_1 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = 0.$$

Thus the vectors are linearly independent. So,

$$\text{Im}(f) = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Basis, $\mathcal{B}_{\text{Im}} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$

3. Rank and isomorphisme

   - **Injectivity:** Since $\text{Ker}(f) \neq \{0_{V=\mathbb{R}^3}\}$, f is NOT injective.
   - **Surjectivity:** Since $\underbrace{\text{Rank}(f)}_{\dim(\text{Im}(f))} = 2 < 3 = \dim(\mathbb{R}^3)$, f is NOT surjective.
   - **Bijectivity:** Not bijective (neither injective nor surjective).

   Isomorphism: Not an isomorphism.

   Rank, $\underbrace{\text{Rank}(f)}_{\dim(\text{Im}(f))} = 2.$

4. Proof that $\mathbb{R}^3 = \text{Ker}(f) \oplus \text{Im}(f)$.

We verify the conditions of Criterion (1.1).

Method 1

Step 1: Trivial intersection. Let

$$\mathbf{v} \in \text{Ker}(f) \cap \text{Im}(f) = \left\{ \mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \mathbf{v} \in \text{Ker}(f), \quad \text{and} \quad \mathbf{v} \in \text{Im}(f) \right\}.$$

Then,

$$\mathbf{v} \in \text{Ker}(f) \Rightarrow \mathbf{v} = x \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ x \\ -x \end{pmatrix} \quad \text{for some } x \in \mathbb{R}.$$

$$\mathbf{v} \in \text{Im}(f) \Rightarrow \mathbf{v} = y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} z \\ y+z \\ y \end{pmatrix} \quad \text{for some } y, z \in \mathbb{R}.$$

Equating components,

$$\begin{cases} 0 = z & (1) \\ x = y + z & (2) \\ -x = y & (3) \end{cases}$$

- From equation (1), $z = 0$;
- From equation (2), $x = y + 0 = y$;
- From equation (3), $-x = y$.

Thus $x = y$ and $-x = y$, so $x = -x \Rightarrow x = 0$, hence $y = 0$.

Thus $\mathbf{v} = 0_{V=\mathbb{R}^3}$, so $\text{Ker}(f) \cap \text{Im}(f) = \{0_{V=\mathbb{R}^3}\}$.

Step 2: Dimension sum.

$$\underbrace{\text{Nullity}(f)}_{\dim(\text{Ker}(f))} + \underbrace{\text{Rank}(f)}_{\dim(\text{Im}(f))} = 1 + 2 = 3 = \dim(\mathbb{R}^3).$$

By Criterion 1.1, we conclude $\mathbb{R}^3 = \text{Ker}(f) \oplus \text{Im}(f)$.

Method 2

We verify the conditions of Definition (1.7).

Step 1: Show $\mathbb{R}^3 = \text{Ker}(f) + \text{Im}(f)$.

We have,

$$\text{Ker}(f) + \text{Im}(f) = \text{Span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Linear independence check

For $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$,

$$\lambda_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \lambda_3 = 0, & (1) \\ -\lambda_1 + \lambda_2 + \lambda_3 = 0, & (2) \\ \lambda_1 + \lambda_2 = 0. & (3) \end{cases}$$

1. From (1) $\lambda_3 = 0$;
2. Substitute into (2): $-\lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_2 = \lambda_1$;
3. Substitute into (3): $\lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_1 + \lambda_1 = 0 \Rightarrow 2\lambda_1 = 0 \Rightarrow \lambda_1 = 0$;

So the vectors are linearly independent.

Then the basis of $\text{Ker}(f) + \text{Im}(f)$ is,

$$\mathcal{B}_{\text{Ker(f)+Im(f)}} = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

and therefore

$$\dim(\text{Ker}(f) + \text{Im}(f)) = 3.$$

Since $\mathrm{Ker}(f) + \mathrm{Im}(f) \subseteq \mathbb{R}^3$ and both have dimension 3,

$$\mathbb{R}^3 = \mathrm{Ker}(f) + \mathrm{Im}(f).$$

Step 2: Show $\mathrm{Ker}(f) \cap \mathrm{Im}(f) = \{\mathbf{0}_{V=\mathbb{R}^3}\}$.

Using the dimension formula,

$$\dim(\mathrm{Ker}(f) + \mathrm{Im}(f)) = \underbrace{\mathrm{Nullity}(f)}_{\dim(\mathrm{Ker}(f))} + \underbrace{\mathrm{Rank}(f)}_{\dim(\mathrm{Im}(f))} - \dim(\mathrm{Ker}(f) \cap \mathrm{Im}(f)).$$

Substitute known values,

$$3 = 1 + 2 - \dim(\mathrm{Ker}(f) \cap \mathrm{Im}(f)).$$

Thus,

$$\dim(\mathrm{Ker}(f) \cap \mathrm{Im}(f)) = 0 \Rightarrow \mathrm{Ker}(f) \cap \mathrm{Im}(f) = \{\mathbf{0}_{V=\mathbb{R}^3}\}.$$

So

$$\mathbb{R}^3 = \mathrm{Ker}(f) \oplus \mathrm{Im}(f).$$

□

Exercise 1.4:
Let V, W be vector spaces, $\{v_1, v_2, v_3\}$ a basis of V, and $\{w_1, w_2, w_3\} \subset W$. Let $f : V \to W$ be linear with $f(v_i) = w_i$ for $i = 1, 2, 3$.

- Prove that f is injective $\iff$ $\{w_1, w_2, w_3\}$ is linearly independent.

Solution:
We prove both directions using Lemma 1.1 and Definition 1.2.

- **($\Rightarrow$) Assume f is injective.**

  Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that,

  $$\lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 = 0_W.$$

  Since f is linear and $f(v_i) = w_i$, we have,

  $$\begin{aligned} f(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) &= \lambda_1 f(v_1) + \lambda_2 f(v_2) + \lambda_3 f(v_3) \\ &= \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 = 0_W. \end{aligned}$$

  Thus $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \in \mathrm{Ker}(f)$.

  By Lemma 1.1, injectivity of f implies $\mathrm{Ker}(f) = \{0_V\}$, so,

  $$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0_V.$$

Since $\{v_1, v_2, v_3\}$ is a basis of V, it is linearly independent. Therefore, $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Thus $\{w_1, w_2, w_3\}$ is linearly independent by Definition 1.2.

- **($\Leftarrow$) Assume $\{w_1, w_2, w_3\}$ is linearly independent.**

  Let $v \in \mathrm{Ker}(f)$. Since $\{v_1, v_2, v_3\}$ is a basis of V, we can write,
  $$v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 \quad \text{for some } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

  Then,
  $$f(v) = \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 = 0_W.$$

  Since $\{w_1, w_2, w_3\}$ is linearly independent, we must have $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus $v = 0_V$.

  Therefore $\mathrm{Ker}(f) = \{0_V\}$, and by Lemma 1.1, f is injective.

$\square$

2. Solutions of Tutorial Exercises (2): A. LESLOUS A. AYACHI

2.1. Basic Definitions

Definition 2.1 (Matrix):
Let m, n be positive integers. An $m \times n$ matrix over a field $\mathbb{F}$ (typically $\mathbb{R}$ or $\mathbb{C}$) is a rectangular array of elements from $\mathbb{F}$ arranged in m rows and n columns,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

We denote by a_{ij} the entry in the i-th row and j-th column. The set of all $m \times n$ matrices over $\mathbb{F}$ is denoted by $\mathcal{M}_{m\times n}(\mathbb{F})$.

***Remark* 2.1 (Matrix Equality):**
Two matrices A and B are equal if and only if,

- They have the same dimensions $(m \times n)$.
- Their corresponding entries are equal, $a_{ij} = b_{ij}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Definition 2.2 (Matrix Addition):
Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$ be matrices in $\mathcal{M}_{m\times n}(\mathbb{F})$. Their sum $A + B$ is defined as,

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}. \tag{2.1}$$

Definition 2.3 (Scalar Multiplication):

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ be a matrix in $\mathcal{M}_{m\times n}(\mathbb{F})$ and let $\lambda \in \mathbb{F}$. The scalar product λA is defined as,

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{pmatrix}. \tag{2.2}$$

Definition 2.4 (Matrix Multiplication):

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathcal{M}_{m\times n}(\mathbb{F})$ and $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix} \in \mathcal{M}_{n\times p}(\mathbb{F})$. Their product AB is defined as the matrix in $\mathcal{M}_{m\times p}(\mathbb{F})$ whose (i, j)-entry is $\sum_{k=1}^{n} a_{ik} b_{kj}$,

$$AB = \begin{pmatrix} \sum_{k=1}^{n} a_{1k} b_{k1} & \sum_{k=1}^{n} a_{1k} b_{k2} & \cdots & \sum_{k=1}^{n} a_{1k} b_{kp} \\ \sum_{k=1}^{n} a_{2k} b_{k1} & \sum_{k=1}^{n} a_{2k} b_{k2} & \cdots & \sum_{k=1}^{n} a_{2k} b_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk} b_{k1} & \sum_{k=1}^{n} a_{mk} b_{k2} & \cdots & \sum_{k=1}^{n} a_{mk} b_{kp} \end{pmatrix}. \tag{2.3}$$

Note: The number of columns of A must equal the number of rows of B.

Caution: Matrix multiplication is not commutative in general.

Definition 2.5 (Transpose of a Matrix):

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathcal{M}_{m\times n}(\mathbb{F})$. The transpose of A, denoted A^T, is the $n \times m$ matrix,

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}. \tag{2.4}$$

Definition 2.6 (Square Matrix):

A matrix A is square if it has the same number of rows and columns, i.e., $A \in \mathcal{M}_{n\times n}(\mathbb{F})$. We denote $\mathcal{M}_n(\mathbb{F}) = \mathcal{M}_{n\times n}(\mathbb{F})$.

Definition 2.7 (Diagonal Matrix):

A square matrix $D = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix} \in \mathcal{M}_n(\mathbb{F})$ is diagonal if $d_{ij} = 0$ for all $i \neq j$.

Definition 2.8 (Identity Matrix):
The $n \times n$ identity matrix I_n is the diagonal matrix with 1's on the diagonal and 0's elsewhere,

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{2.5}$$

Definition 2.9 (Symmetric Matrix):
A square matrix $A \in \mathcal{M}_n(\mathbb{F})$ is symmetric if $A^T = A$.

Definition 2.10 (Upper/Lower Triangular Matrix):
A square matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in \mathcal{M}_n(\mathbb{F})$ is,

- Upper triangular if $a_{ij} = 0$ for all $i > j$
- Lower triangular if $a_{ij} = 0$ for all $i < j$

Definition 2.11 (Minor):
Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be an $n \times n$ matrix. The (i, j)-minor M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting the i-th row and j-th column from A.

Definition 2.12 (Cofactor Matrix):
For an $n \times n$ matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, the cofactor matrix $\mathrm{Cof}(A)$ is defined as,

$$\mathrm{Cof}(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix} = \begin{pmatrix} (-1)^{1+1}M_{11} & (-1)^{1+2}M_{12} & \cdots & (-1)^{1+n}M_{1n} \\ (-1)^{2+1}M_{21} & (-1)^{2+2}M_{22} & \cdots & (-1)^{2+n}M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}M_{n1} & (-1)^{n+2}M_{n2} & \cdots & (-1)^{n+n}M_{nn} \end{pmatrix}, \tag{2.6}$$

where each element $C_{ij} = (-1)^{i+j}M_{ij}$, with M_{ij} being the (i, j)-minor of A.

Theorem 2.1 (Cofactor Expansion):
For any fixed row i,

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}. \tag{2.7}$$

For any fixed column j,

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}.$$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\det(A) = ad - bc. \tag{2.8}$$

Theorem 2.2 (Basic Properties):
Let $A, B \in \mathcal{M}_n(\mathbb{F})$ and $\lambda \in \mathbb{F}$,

1. $\det(I_n) = 1$ where $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

2. $\det(A^T) = \det(A)$

3. $\det(AB) = \det(A)\det(B)$

4. $\det(\lambda A) = \lambda^n \det(A)$

5. If A is triangular, $\det(A)$ is the product of its diagonal entries

Definition 2.13 (Invertible Matrix):
A square matrix $A \in \mathcal{M}_n(\mathbb{F})$ is invertible (or non-singular) if there exists a matrix $B \in \mathcal{M}_n(\mathbb{F})$ such that,

$$AB = BA = I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{2.9}$$

If such B exists, it is unique and is called the inverse of A, denoted A^{-1}.

Theorem 2.3 (Properties of Matrix Inverses):
Let $A, B \in \mathcal{M}_n(\mathbb{F})$ be invertible and $\lambda \in \mathbb{F} \setminus \{0\}$,

1. $(A^{-1})^{-1} = A$

2. $(AB)^{-1} = B^{-1}A^{-1}$

3. $(A^T)^{-1} = (A^{-1})^T$

4. $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

5. $\det(A^{-1}) = \frac{1}{\det(A)}$

Theorem 2.4 (Characterization of Invertibility):
For $A \in \mathcal{M}_n(\mathbb{F})$, the following are equivalent,

1. A is invertible

2. $\det(A) \neq 0$

3. The rows (or columns) of A are linearly independent

4. A has rank n

Theorem 2.5 (Adjugate Formula):
For $A \in \mathcal{M}_n(\mathbb{F})$, let C_{ij} be the (i, j)-cofactor. The adjugate matrix is,

$$\operatorname{adj}(A) = (\operatorname{Cof}(A))^T = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}. \tag{2.10}$$

If $\det(A) \neq 0$, then,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A). \tag{2.11}$$

- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(A) = ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{2.12}$$

- For a 3×3 matrix A with $\det(A) \neq 0$,

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix},$$

  where $\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$.

Definition 2.14 (Basis and Coordinates):
A basis of a vector space V is a linearly independent set that spans V. If $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then every vector $v \in V$ can be uniquely expressed as,

$$v = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

The scalars $x_1, x_2, \ldots, x_n$ are called the coordinates of v relative to $\mathcal{B}$, and we write,

$$[v]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Definition 2.15 (Matrix of a Linear map):
Let $f : V \to W$ be a linear map, $\mathcal{B}_V = \{v_1, v_2, \ldots, v_n\}$ a basis for V, and $\mathcal{B}_W = \{w_1, w_2, \ldots, w_m\}$ a basis for W. The matrix of f relative to these bases, denoted $[f]_{\mathcal{B}_V}^{\mathcal{B}_W}$, is the $m \times n$ matrix whose j-th column is $[f(v_j)]_{\mathcal{B}_W}$.

Theorem 2.6 (Matrix Representation Theorem):
For any linear map $f : V \to W$ with matrix $A = [f]_{\mathcal{B}_V}^{\mathcal{B}_W}$, we have,

$$[f(v)]_{\mathcal{B}_W} = A[v]_{\mathcal{B}_V} \quad \text{for all } v \in V. \tag{2.13}$$

Definition 2.16 (Change-of-Basis Matrix):
Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{B}' = \{v_1', v_2', \ldots, v_n'\}$ be two bases for V. The change-of-basis matrix from $\mathcal{B}'$ to $\mathcal{B}$ is,

$$P = [\mathrm{id}]_{\mathcal{B}'}^{\mathcal{B}} = \begin{pmatrix} | & | & & | \\ [v_1']_{\mathcal{B}} & [v_2']_{\mathcal{B}} & \cdots & [v_n']_{\mathcal{B}} \\ | & | & & | \end{pmatrix}, \tag{2.14}$$

where $\mathrm{id} : V \to V$ is the identity map.

Theorem 2.7 (Change of Basis for Vectors):
For any $v \in V$,

$$[v]_{\mathcal{B}} = P[v]_{\mathcal{B}'}$$

and

$$[v]_{\mathcal{B}'} = P^{-1}[v]_{\mathcal{B}}.$$

Theorem 2.8 (Change of Basis for Linear maps):
Let $f : V \to V$ be linear, and let $\mathcal{B}$ and $\mathcal{B}'$ be bases for V. If P is the change-of-basis matrix from $\mathcal{B}'$ to $\mathcal{B}$, then,

$$[f]_{\mathcal{B}'}^{\mathcal{B}'} = P^{-1}[f]_{\mathcal{B}}^{\mathcal{B}}P. \tag{2.15}$$

Exercise 2.1:
Let A, B and C be the following matrices,

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ 0 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 0 & 5 \\ -2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -3 \\ 2 & 1 \\ 8 & 0 \end{pmatrix}.$$

Calculate,

$$B + C, \quad B - C, \quad 2A, \quad A^T, \quad AB, \quad A^2,$$

where A^T denotes the transpose of A.

Solution:

1. Calculate $B + C$

   By the definition of matrix addition in (2.1), we compute

$$B + C = \begin{pmatrix} 1 & 2 \\ 0 & 5 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -3 \\ 2 & 1 \\ 8 & 0 \end{pmatrix} = \begin{pmatrix} 1+0 & 2+(-3) \\ 0+2 & 5+1 \\ -2+8 & 1+0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 6 \\ 6 & 1 \end{pmatrix}.$$

2. Calculate $B - C$

   Using the definition of matrix subtraction, which follows from (2.1) and scalar multiplication, we obtain

$$B - C = \begin{pmatrix} 1 & 2 \\ 0 & 5 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -3 \\ 2 & 1 \\ 8 & 0 \end{pmatrix} = \begin{pmatrix} 1-0 & 2-(-3) \\ 0-2 & 5-1 \\ -2-8 & 1-0 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -2 & 4 \\ -10 & 1 \end{pmatrix}.$$

3. Calculate $2A$

   By the definition of scalar multiplication in (2.2), we have

$$2A = 2 \times \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -2 \\ -6 & 8 & -6 \\ 0 & 4 & 2 \end{pmatrix}.$$

4. Calculate A^T (transpose of A)

   According to the definition of the transpose in (2.4),

$$A^T = \begin{pmatrix} 0 & -3 & 0 \\ 1 & 4 & 2 \\ -1 & -3 & 1 \end{pmatrix}.$$

5. Calculate AB

   Since A is 3×3 and B is 3×2, the product AB is defined by (2.3) and yields a 3×2 matrix.

$$\begin{aligned} AB &= \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 5 \\ -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \cdot 1 + 1 \cdot 0 + (-1) \cdot (-2) & 0 \cdot 2 + 1 \cdot 5 + (-1) \cdot 1 \\ (-3) \cdot 1 + 4 \cdot 0 + (-3) \cdot (-2) & (-3) \cdot 2 + 4 \cdot 5 + (-3) \cdot 1 \\ 0 \cdot 1 + 2 \cdot 0 + 1 \cdot (-2) & 0 \cdot 2 + 2 \cdot 5 + 1 \cdot 1 \end{pmatrix} \\ &= \begin{pmatrix} 0+0+2 & 0+5-1 \\ -3+0+6 & -6+20-3 \\ 0+0-2 & 0+10+1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 11 \\ -2 & 11 \end{pmatrix}. \end{aligned}$$

6. Calculate A^2

   By repeated application of matrix multiplication, we compute

$$\begin{aligned} A^2 = A \cdot A &= \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ 0 & 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot (-3) & 1 \cdot 4 + (-1) \cdot 2 & 1 \cdot (-3) + (-1) \cdot 1 \\ 4 \cdot (-3) & (-3) \cdot 1 + 4 \cdot 4 + (-3) \cdot 2 & (-3) \cdot (-1) + 4 \cdot (-3) + (-3) \cdot 1 \\ 2 \cdot (-3) & 2 \cdot 4 + 1 \cdot 2 & 2 \cdot (-3) + 1 \cdot 1 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 2 & -4 \\ -12 & 7 & -12 \\ -6 & 10 & -5 \end{pmatrix}. \end{aligned}$$

□

Exercise 2.2:
Calculate the determinants of the following matrices,

$$A = \begin{pmatrix} 7 & 11 \\ -8 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} 10 & 0 & -5 & 15 \\ -2 & 7 & 3 & 0 \\ 8 & 14 & 0 & 2 \\ 0 & -21 & 1 & -1 \end{pmatrix}.$$

Solution:

1. Determinant of $A = \begin{pmatrix} 7 & 11 \\ -8 & 4 \end{pmatrix}$

   Using formula (2.8) for the determinant of a 2×2 matrix,

$$\det(A) = 7 \cdot 4 - 11 \cdot (-8) = 28 + 88 = 116.$$

2. Determinant of $B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{pmatrix}$

   We compute using cofactor expansion along the first row as stated in (2.7),

$$\begin{aligned} \det(B) &= 1 \cdot \det\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 3 & 5 \\ 5 & 7 \end{pmatrix} + 2 \cdot \det\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \\ &= 1 \cdot (4 \cdot 7 - 5 \cdot 6) + 2 \cdot (3 \cdot 6 - 4 \cdot 5) \\ &= 1 \cdot (28 - 30) + 2 \cdot (18 - 20) = 1 \cdot (-2) + 2 \cdot (-2) \\ &= -2 - 4 = -6. \end{aligned}$$

3. Determinant of $C = \begin{pmatrix} 10 & 0 & -5 & 15 \\ -2 & 7 & 3 & 0 \\ 8 & 14 & 0 & 2 \\ 0 & -21 & 1 & -1 \end{pmatrix}$

   We expand along the first row using (2.7),

$$\det(C) = 10 \cdot C_{11} + 0 \cdot C_{12} + (-5) \cdot C_{13} + 15 \cdot C_{14}, \tag{2.16}$$

   where C_{ij} are cofactors defined in (2.6). We compute each required cofactor,

$$\begin{aligned} C_{11} &= (-1)^{1+1} \det\begin{pmatrix} 7 & 3 & 0 \\ 14 & 0 & 2 \\ -21 & 1 & -1 \end{pmatrix} = \det\begin{pmatrix} 7 & 3 & 0 \\ 14 & 0 & 2 \\ -21 & 1 & -1 \end{pmatrix} \\ &= 7 \cdot \det\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} - 3 \cdot \det\begin{pmatrix} 14 & 2 \\ -21 & -1 \end{pmatrix} + 0 \cdot \det\begin{pmatrix} 14 & 0 \\ -21 & 1 \end{pmatrix} \\ &= 7 \cdot (0 \cdot (-1) - 2 \cdot 1) - 3 \cdot (14 \cdot (-1) - 2 \cdot (-21)) \\ &= 7 \cdot (-2) - 3 \cdot (-14 + 42) = -14 - 3 \cdot 28 = -14 - 84 = -98, \end{aligned} \tag{2.17}$$

$$\begin{aligned} C_{13} &= (-1)^{1+3} \det\begin{pmatrix} -2 & 7 & 0 \\ 8 & 14 & 2 \\ 0 & -21 & -1 \end{pmatrix} = \det\begin{pmatrix} -2 & 7 & 0 \\ 8 & 14 & 2 \\ 0 & -21 & -1 \end{pmatrix} \\ &= -2 \cdot \det\begin{pmatrix} 14 & 2 \\ -21 & -1 \end{pmatrix} - 7 \cdot \det\begin{pmatrix} 8 & 2 \\ 0 & -1 \end{pmatrix} + 0 \cdot \det\begin{pmatrix} 8 & 14 \\ 0 & -21 \end{pmatrix} \\ &= -2 \cdot (14 \cdot (-1) - 2 \cdot (-21)) - 7 \cdot (8 \cdot (-1) - 2 \cdot 0) \\ &= -2 \cdot (-14 + 42) - 7 \cdot (-8) = -2 \cdot 28 + 56 = -56 + 56 = 0, \end{aligned} \tag{2.18}$$

$$\begin{aligned} C_{14} &= (-1)^{1+4} \det\begin{pmatrix} -2 & 7 & 3 \\ 8 & 14 & 0 \\ 0 & -21 & 1 \end{pmatrix} = -\det\begin{pmatrix} -2 & 7 & 3 \\ 8 & 14 & 0 \\ 0 & -21 & 1 \end{pmatrix} \\ &= -\left[-2 \cdot \det\begin{pmatrix} 14 & 0 \\ -21 & 1 \end{pmatrix} - 7 \cdot \det\begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix} + 3 \cdot \det\begin{pmatrix} 8 & 14 \\ 0 & -21 \end{pmatrix}\right] \\ &= -[-2 \cdot (14 \cdot 1 - 0 \cdot (-21)) - 7 \cdot (8 \cdot 1 - 0 \cdot 0) + 3 \cdot (8 \cdot (-21) - 14 \cdot 0)] \\ &= -[-2 \cdot 14 - 7 \cdot 8 + 3 \cdot (-168)] = -[-28 - 56 - 504] = -(-588) = 588. \end{aligned} \tag{2.19}$$

Substituting the values from (2.17), (2.18), and (2.19) into (2.16), we obtain

$$\det(C) = 10 \cdot (-98) + (-5) \cdot 0 + 15 \cdot 588 = -980 + 0 + 8820 = 7840.$$

$\square$

Exercise 2.3:
Calculate the inverse of the following matrices,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ -2 & -2 & -1 \end{pmatrix}.$$

Solution:

1. Inverse of $B = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ -2 & -2 & -1 \end{pmatrix}$

We compute the determinant using cofactor expansion along the first row as in (2.7),

$$\begin{aligned}
\det(B) &= 1 \cdot \det\begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix} - 2 \cdot \det\begin{pmatrix} 1 & -1 \\ -2 & -1 \end{pmatrix} + 1 \cdot \det\begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} \\
&= 1 \cdot (2 \cdot (-1) - (-1) \cdot (-2)) - 2 \cdot (1 \cdot (-1) - (-1) \cdot (-2)) + 1 \cdot (1 \cdot (-2) - 2 \cdot (-2)) \\
&= 1 \cdot (-2 - 2) - 2 \cdot (-1 - 2) + 1 \cdot (-2 + 4) = 1 \cdot (-4) - 2 \cdot (-3) + 1 \cdot 2 \\
&= -4 + 6 + 2 = 4.
\end{aligned}$$

Since $\det(B) \neq 0$, the inverse exists. We compute the matrix of cofactors as defined in (2.6),

$$\begin{aligned}
\mathrm{Cof}(B) &= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} +\det\begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix} & -\det\begin{pmatrix} 1 & -1 \\ -2 & -1 \end{pmatrix} & +\det\begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} \\ -\det\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} & +\det\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} \\ +\det\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & +\det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 2 \cdot (-1) - (-1) \cdot (-2) & -(1 \cdot (-1) - (-1) \cdot (-2)) & 1 \cdot (-2) - 2 \cdot (-2) \\ -(2 \cdot (-1) - 1 \cdot (-2)) & 1 \cdot (-1) - 1 \cdot (-2) & -(1 \cdot (-2) - 2 \cdot (-2)) \\ 2 \cdot (-1) - 1 \cdot 2 & -(1 \cdot (-1) - 1 \cdot 1) & 1 \cdot 2 - 2 \cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} -4 & 3 & 2 \\ 0 & 1 & -2 \\ -4 & 2 & 0 \end{pmatrix}.
\end{aligned}$$

The adjugate matrix, by definition (2.10),

$$\mathrm{adj}(B) = (\mathrm{Cof}(B))^T = \begin{pmatrix} -4 & 0 & -4 \\ 3 & 1 & 2 \\ 2 & -2 & 0 \end{pmatrix}.$$

Finally, applying the adjugate formula (2.11),

$$B^{-1} = \frac{1}{\det(B)}\operatorname{adj}(B) = \frac{1}{4}\begin{pmatrix} -4 & 0 & -4 \\ 3 & 1 & 2 \\ 2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ \frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}.$$

2. Inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$

First, we compute the determinant using (2.8),

$$\det(A) = 2 \cdot (-1) - 1 \cdot 1 = -2 - 1 = -3.$$

Since $\det(A) \neq 0$, the matrix is invertible.

Cofactor matrix,

$$\operatorname{Cof}(A) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} +(-1) & -(1) \\ -(1) & +(2) \end{pmatrix}.$$

Adjugate,

$$\operatorname{adj}(A) = (\operatorname{Cof}(A))^T = \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Inverse,

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}.$$

□

Exercise 2.4:
Consider the matrices,

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -1 \\ 4 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & 3 \end{pmatrix}.$$

(1) Calculate $A + B$, $A - B$, $2A$.

(2) Calculate, if possible, AB, BA, A^2, A^3.

(3) Calculate A^T and B^T.

(4) Calculate A^{-1}.

(5) Calculate $A^3 - 4A^2 + 2A - 17I$; then conclude that A is invertible and give A^{-1}.

Solution:

1. Calculate $A+B$, $A-B$, $2A$

   By the definitions of matrix addition and scalar multiplication,

$$A+B=\begin{pmatrix}2+1 & -1+3 & 2+5\\ 3+1 & 1+2 & -1+4\\ 4+1 & 2+1 & 1+3\end{pmatrix}=\begin{pmatrix}3&2&7\\4&3&3\\5&3&4\end{pmatrix}, \tag{2.20}$$

$$A-B=\begin{pmatrix}2-1 & -1-3 & 2-5\\ 3-1 & 1-2 & -1-4\\ 4-1 & 2-1 & 1-3\end{pmatrix}=\begin{pmatrix}1&-4&-3\\2&-1&-5\\3&1&-2\end{pmatrix}, \tag{2.21}$$

$$2A=\begin{pmatrix}4&-2&4\\6&2&-2\\8&4&2\end{pmatrix}. \tag{2.22}$$

2. Calculate AB, BA, A^2, A^3

   We compute each product using the definition of matrix multiplication (2.3).

   - AB,

$$\begin{aligned}AB&=\begin{pmatrix}2&-1&2\\3&1&-1\\4&2&1\end{pmatrix}\begin{pmatrix}1&3&5\\1&2&4\\1&1&3\end{pmatrix}\\
&=\begin{pmatrix}2\cdot1+(-1)\cdot1+2\cdot1 & 2\cdot3+(-1)\cdot2+2\cdot1 & 2\cdot5+(-1)\cdot4+2\cdot3\\ 3\cdot1+1\cdot1+(-1)\cdot1 & 3\cdot3+1\cdot2+(-1)\cdot1 & 3\cdot5+1\cdot4+(-1)\cdot3\\ 4\cdot1+2\cdot1+1\cdot1 & 4\cdot3+2\cdot2+1\cdot1 & 4\cdot5+2\cdot4+1\cdot3\end{pmatrix}\\
&=\begin{pmatrix}2-1+2 & 6-2+2 & 10-4+6\\ 3+1-1 & 9+2-1 & 15+4-3\\ 4+2+1 & 12+4+1 & 20+8+3\end{pmatrix}=\begin{pmatrix}3&6&12\\3&10&16\\7&17&31\end{pmatrix}.\end{aligned}$$

   - BA,

$$\begin{aligned}BA&=\begin{pmatrix}1&3&5\\1&2&4\\1&1&3\end{pmatrix}\begin{pmatrix}2&-1&2\\3&1&-1\\4&2&1\end{pmatrix}\\
&=\begin{pmatrix}1\cdot2+3\cdot3+5\cdot4 & 1\cdot(-1)+3\cdot1+5\cdot2 & 1\cdot2+3\cdot(-1)+5\cdot1\\ 1\cdot2+2\cdot3+4\cdot4 & 1\cdot(-1)+2\cdot1+4\cdot2 & 1\cdot2+2\cdot(-1)+4\cdot1\\ 1\cdot2+1\cdot3+3\cdot4 & 1\cdot(-1)+1\cdot1+3\cdot2 & 1\cdot2+1\cdot(-1)+3\cdot1\end{pmatrix}\\
&=\begin{pmatrix}2+9+20 & -1+3+10 & 2-3+5\\ 2+6+16 & -1+2+8 & 2-2+4\\ 2+3+12 & -1+1+6 & 2-1+3\end{pmatrix}=\begin{pmatrix}31&12&4\\24&9&4\\17&6&4\end{pmatrix}\ (\neq AB).\end{aligned}$$

- $A^2 = A \cdot A$,

$$A^2 = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -1 \\ 4 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\cdot 2 + (-1)\cdot 3 + 2\cdot 4 & 2\cdot(-1) + (-1)\cdot 1 + 2\cdot 2 & 2\cdot 2 + (-1)\cdot(-1) + 2\cdot 1 \\ 3\cdot 2 + 1\cdot 3 + (-1)\cdot 4 & 3\cdot(-1) + 1\cdot 1 + (-1)\cdot 2 & 3\cdot 2 + 1\cdot(-1) + (-1)\cdot 1 \\ 4\cdot 2 + 2\cdot 3 + 1\cdot 4 & 4\cdot(-1) + 2\cdot 1 + 1\cdot 2 & 4\cdot 2 + 2\cdot(-1) + 1\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 - 3 + 8 & -2 - 1 + 4 & 4 + 1 + 2 \\ 6 + 3 - 4 & -3 + 1 - 2 & 6 - 1 - 1 \\ 8 + 6 + 4 & -4 + 2 + 2 & 8 - 2 + 1 \end{pmatrix} = \begin{pmatrix} 9 & 1 & 7 \\ 5 & -4 & 4 \\ 18 & 0 & 7 \end{pmatrix}. \tag{2.23}$$

- $A^3 = A^2 \cdot A$,

$$A^3 = \begin{pmatrix} 9 & 1 & 7 \\ 5 & -4 & 4 \\ 18 & 0 & 7 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -1 \\ 4 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9\cdot 2 + 1\cdot 3 + 7\cdot 4 & 9\cdot(-1) + 1\cdot 1 + 7\cdot 2 & 9\cdot 2 + 1\cdot(-1) + 7\cdot 1 \\ 5\cdot 2 + (-4)\cdot 3 + 4\cdot 4 & 5\cdot(-1) + (-4)\cdot 1 + 4\cdot 2 & 5\cdot 2 + (-4)\cdot(-1) + 4\cdot 1 \\ 18\cdot 2 + 0\cdot 3 + 7\cdot 4 & 18\cdot(-1) + 0\cdot 1 + 7\cdot 2 & 18\cdot 2 + 0\cdot(-1) + 7\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 18 + 3 + 28 & -9 + 1 + 14 & 18 - 1 + 7 \\ 10 - 12 + 16 & -5 - 4 + 8 & 10 + 4 + 4 \\ 36 + 0 + 28 & -18 + 0 + 14 & 36 + 0 + 7 \end{pmatrix} = \begin{pmatrix} 49 & 6 & 24 \\ 14 & -1 & 18 \\ 64 & -4 & 43 \end{pmatrix}. \tag{2.24}$$

3. Calculate A^T and B^T

By the definition of the transpose (2.4),

$$A^T = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}, \quad B^T = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{pmatrix}.$$

4. Calculate A^{-1}

First, compute $\det(A)$ using cofactor expansion along the first row as in (2.7),

$$\begin{aligned} \det(A) &= 2\cdot\det\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} - (-1)\cdot\det\begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix} + 2\cdot\det\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \\ &= 2\cdot(1\cdot 1 - (-1)\cdot 2) + 1\cdot(3\cdot 1 - (-1)\cdot 4) + 2\cdot(3\cdot 2 - 1\cdot 4) \\ &= 2\cdot(1 + 2) + 1\cdot(3 + 4) + 2\cdot(6 - 4) \\ &= 2\cdot 3 + 1\cdot 7 + 2\cdot 2 = 6 + 7 + 4 = 17 \neq 0. \end{aligned}$$

Now compute the matrix of cofactors as defined in (2.6),

$$\begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -1 \\ 4 & 2 & 1 \end{pmatrix}$$

$$\begin{aligned}
\mathrm{Cof}(A) &= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \\
&= \begin{pmatrix} +\det\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} & -\det\begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix} & +\det\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \\ -\det\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} & +\det\begin{pmatrix} 2 & 2 \\ 4 & 1 \end{pmatrix} & -\det\begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \\ +\det\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} & -\det\begin{pmatrix} 2 & 2 \\ 3 & -1 \end{pmatrix} & +\det\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} 1\cdot 1-(-1)\cdot 2 & -(3\cdot 1-(-1)\cdot 4) & 3\cdot 2-1\cdot 4 \\ -((-1)\cdot 1-2\cdot 2) & 2\cdot 1-2\cdot 4 & -(2\cdot 2-(-1)\cdot 4) \\ (-1)\cdot(-1)-2\cdot 1 & -(2\cdot(-1)-2\cdot 3) & 2\cdot 1-(-1)\cdot 3 \end{pmatrix} \\
&= \begin{pmatrix} 3 & -7 & 2 \\ 5 & -6 & -8 \\ -1 & 8 & 5 \end{pmatrix}.
\end{aligned}$$

The adjugate matrix, by (2.10), is,

$$\mathrm{adj}(A) = \mathrm{Cof}(A)^T = \begin{pmatrix} 3 & 5 & -1 \\ -7 & -6 & 8 \\ 2 & -8 & 5 \end{pmatrix}.$$

Finally, by the adjugate formula (2.11),

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A) = \frac{1}{17}\begin{pmatrix} 3 & 5 & -1 \\ -7 & -6 & 8 \\ 2 & -8 & 5 \end{pmatrix}. \tag{2.25}$$

5. Calculate $A^3 - 4A^2 + 2A - 17I$

From (2.24), (2.23), (2.20) and (2.5) we have,

$$\begin{aligned}
4A^2 &= 4\cdot\begin{pmatrix} 9 & 1 & 7 \\ 5 & -4 & 4 \\ 18 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 36 & 4 & 28 \\ 20 & -16 & 16 \\ 72 & 0 & 28 \end{pmatrix}, \\
2A &= \begin{pmatrix} 4 & -2 & 4 \\ 6 & 2 & -2 \\ 8 & 4 & 2 \end{pmatrix}, \\
17I &= \begin{pmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{pmatrix}.
\end{aligned}$$

Now,

$$A^3 - 4A^2 + 2A - 17I = \begin{pmatrix} 49 & 6 & 24 \\ 14 & -1 & 18 \\ 64 & -4 & 43 \end{pmatrix} - \begin{pmatrix} 36 & 4 & 28 \\ 20 & -16 & 16 \\ 72 & 0 & 28 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 4 \\ 6 & 2 & -2 \\ 8 & 4 & 2 \end{pmatrix}$$

$$
\begin{aligned}
&-\begin{pmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{pmatrix} \\
&= \begin{pmatrix} 49-36+4-17 & 6-4-2-0 & 24-28+4-0 \\ 14-20+6-0 & -1+16+2-17 & 18-16-2-0 \\ 64-72+8-0 & -4-0+4-0 & 43-28+2-17 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

Thus, $A^3 - 4A^2 + 2A - 17I = 0$.

This implies that $A^3 = 4A^2 - 2A + 17I$. Multiply both sides by A^{-1} (if it exists) to get $A^2 = 4A - 2I + 17A^{-1}$, so

$$17A^{-1} = A^2 - 4A + 2I.$$

Thus,

$$
\begin{aligned}
A^{-1} &= \frac{1}{17}(A^2 - 4A + 2I) \\
&= \frac{1}{17}\left(\begin{pmatrix} 9 & 1 & 7 \\ 5 & -4 & 4 \\ 18 & 0 & 7 \end{pmatrix} - 4\begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & -1 \\ 4 & 2 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\
&= \frac{1}{17}\begin{pmatrix} 9-8+2 & 1+4 & 7-8 \\ 5-12 & -4-4+2 & 4+4 \\ 18-16 & -8 & 7-4+2 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 3 & 5 & -1 \\ -7 & -6 & 8 \\ 2 & -8 & 5 \end{pmatrix}
\end{aligned}
$$

which is consistent with our previous computation in (2.25).

□

Exercise 2.5:
Let,

$$A_\alpha = \begin{pmatrix} 1 & -3 & \alpha \\ \alpha & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}.$$

(1) Calculate, if possible, A_α^2, $2A_\alpha$, and $\det\left(A_\alpha^2\right)$.

(2) For $\alpha = 6$, find real numbers a and b such that,

$$A_6^2 = aA_6 + bI_3, \quad \text{where } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(3) Deduce that A_6 is invertible and give A_6^{-1}.

(4) If $\alpha = 6 \pm \frac{2\sqrt{3}}{3}$, A_α invertible? Explain your conclusion.

Solution:

1. Compute A_α^2, $2A_\alpha$, and $\det(A_\alpha^2)$

   First, compute A_α^2 using matrix multiplication (2.3),

$$\begin{aligned} A_\alpha^2 &= \begin{pmatrix} 1 & -3 & \alpha \\ \alpha & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -3 & \alpha \\ \alpha & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1-3\alpha+3\alpha & -3+24-3\alpha & \alpha-36+4\alpha \\ \alpha-8\alpha+36 & -3\alpha+64-36 & \alpha^2-96+48 \\ 3-3\alpha+12 & -9+24-12 & 3\alpha-36+16 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 21-3\alpha & 5\alpha-36 \\ 36-7\alpha & 28-3\alpha & \alpha^2-48 \\ 15-3\alpha & 3 & 3\alpha-20 \end{pmatrix}. \end{aligned} \tag{2.26}$$

   Next, by scalar multiplication (2.2),

$$2A_\alpha = \begin{pmatrix} 2 & -6 & 2\alpha \\ 2\alpha & -16 & 24 \\ 6 & -6 & 8 \end{pmatrix}.$$

   Now, by the property $\det(AB) = \det(A)\det(B)$ from Theorem 2.2, we have $\det(A_\alpha^2) = (\det(A_\alpha))^2$. So compute $\det(A_\alpha)$ first using cofactor expansion,

$$\begin{aligned} \det(A_\alpha) &= 1\cdot\det\begin{pmatrix} -8 & 12 \\ -3 & 4 \end{pmatrix} - (-3)\cdot\det\begin{pmatrix} \alpha & 12 \\ 3 & 4 \end{pmatrix} + \alpha\cdot\det\begin{pmatrix} \alpha & -8 \\ 3 & -3 \end{pmatrix} \\ &= 1\cdot((-8)\cdot 4 - 12\cdot(-3)) + 3\cdot(\alpha\cdot 4 - 12\cdot 3) + \alpha\cdot(\alpha\cdot(-3) - (-8)\cdot 3) \\ &= 1\cdot(-32+36) + 3\cdot(4\alpha-36) + \alpha\cdot(-3\alpha+24) \\ &= 4 + 12\alpha - 108 - 3\alpha^2 + 24\alpha = -3\alpha^2 + 36\alpha - 104. \end{aligned} \tag{2.27}$$

   Thus,

$$\det(A_\alpha^2) = \left(-3\alpha^2 + 36\alpha - 104\right)^2$$

2. For $\alpha = 6$, find a, b such that $A_6^2 = aA_6 + bI$

   Substitute $\alpha = 6$ into (2.26) and (2.27),

$$\begin{aligned} A_6 &= \begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix}, \\ A_6^2 &= \begin{pmatrix} 1 & 21-18 & 30-36 \\ 36-42 & 28-18 & 36-48 \\ 15-18 & 3 & 18-20 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -6 \\ -6 & 10 & -12 \\ -3 & 3 & -2 \end{pmatrix}. \end{aligned} \tag{2.28}$$

We want a, b such that

$$A_6^2 = aA_6 + bI = a\begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} + b\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a+b & -3a & 6a \\ 6a & -8a+b & 12a \\ 3a & -3a & 4a+b \end{pmatrix}.$$

Equating with A_6^2 from (2.28) gives the system,

$$-3a = 3 \quad (1,2) \Rightarrow a = -1$$

$$-8a + b = 10 \quad (2,2) \Rightarrow -8(-1) + b = 10 \Rightarrow 8 + b = 10 \Rightarrow b = 2$$

Thus, $a = -1, b = 2$. So,

$$A_6^2 = -A_6 + 2I.$$

3. Deduce that A_6 is invertible and find A_6^{-1}

   From the equation $A_6^2 = -A_6 + 2I$, rearrange,

$$A_6^2 + A_6 = 2I \quad \Rightarrow \quad A_6(A_6 + I) = 2I.$$

   Thus,

$$A_6 \cdot \frac{1}{2}(A_6 + I) = I.$$

   By the definition (2.13) (of the inverse), this shows that A_6 is invertible with inverse

$$A_6^{-1} = \frac{1}{2}(A_6 + I) = \frac{1}{2}\left(\begin{pmatrix} 1 & -3 & 6 \\ 6 & -8 & 12 \\ 3 & -3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 2 & -3 & 6 \\ 6 & -7 & 12 \\ 3 & -3 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{3}{2} & 3 \\ 3 & -\frac{7}{2} & 6 \\ \frac{3}{2} & -\frac{3}{2} & \frac{5}{2} \end{pmatrix}.$$

4. We have

$$\det(A_\alpha) = -3\alpha^2 + 36\alpha - 10.$$

   For $\alpha = 6 \pm \frac{2\sqrt{3}}{3}$,

$$\det(A_\alpha) = 0,$$

   then A_α is not invertible, and A_α^{-1}does not exist.

□

Exercise 2.6:
Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by,

$$f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + y \\ x + z + 2y \end{pmatrix},$$

with bases,

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}, \quad \mathcal{B}' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

- Find the matrix representing f with respect to $\mathcal{B}$ and $\mathcal{B}'$.

Solution:

- Matrix of the linear map f

  Apply f to each vector in $\mathcal{B}$,

$$f\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1+1 \\ 1+0+2\cdot 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$
$$f\left(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 0+1 \\ 0+(-1)+2\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$f\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1+2 \\ 1+1+2\cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

  Now express each image in terms of basis $\mathcal{B}'$. For a vector $\begin{pmatrix} a \\ b \end{pmatrix}$, we want coefficients α, β such that

$$\alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

  This gives the system,

$$\begin{cases} \alpha = a \\ \alpha + \beta = b \end{cases} \Rightarrow \begin{cases} \alpha = a \\ \beta = b - a \end{cases}.$$

  Thus,

  - For $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\alpha = 2, \beta = 3 - 2 = 1$.

  - For $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\alpha = 1, \beta = 1 - 1 = 0$.

  - For $\begin{pmatrix} 3 \\ 6 \end{pmatrix}$, $\alpha = 3, \beta = 6 - 3 = 3$.

Therefore, by Definition (2.13), the matrix representing f with respect to $\mathcal{B}$ and $\mathcal{B}'$ is,

$$[f]_{\mathcal{B}}^{\mathcal{B}'} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 3 \end{pmatrix}.$$

□

Exercise 2.7:
Let $\mathbb{R}_2[X] = \{a_0 + a_1 X + a_2 X^2\}$ be the vector space of real polynomials of degree at most 2, and let $\mathcal{B} = (1, X, X^2)$ be its canonical basis. Consider the map,

$$f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X], \quad f(P) = (1 + X)P'(X).$$

(1) Prove that f is a linear map.

(2) Prove that the matrix of f with respect to $\mathcal{B}$ is,

$$[f]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

(3) Let $\mathcal{B}' = \{1, 1 + X, (1 + X)^2\}$. Find the matrix $[f]_{\mathcal{B}'}^{\mathcal{B}'}$ of f with respect to the basis $\mathcal{B}'$.

Solution:

1. Prove that f is linear

   Let $P, Q \in \mathbb{R}_2[X]$ and $\lambda \in \mathbb{R}$. Then, by the linearity of the derivative,

$$\begin{aligned} f(P + \lambda Q) &= (1 + X)(P + \lambda Q)'(X) = (1 + X)(P' + \lambda Q')(X) \\ &= (1 + X)P'(X) + \lambda(1 + X)Q'(X) = f(P) + \lambda f(Q). \end{aligned}$$

   Thus, f is linear.

2. Matrix of f with respect to the canonical basis $\mathcal{B} = (v_1 = 1, v_2 = X, v_3 = X^2)$

   Apply f to each basis vector,

$$\begin{aligned} f(1) &= (1 + X) \cdot 0 = 0 \\ &= 0 \cdot 1 + 0 \cdot X + 0 \cdot X^2 \\ &= \lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \lambda_3 \cdot v_3 \\ &\Longrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0, \\ f(X) &= (1 + X) \cdot 1 = 1 + X \\ &= 1 \cdot 1 + 1 \cdot X + 0 \cdot X^2 \\ &= \lambda'_1 \cdot v_1 + \lambda'_2 \cdot v_2 + \lambda'_3 \cdot v_3 \\ &\Longrightarrow \lambda'_1 = \lambda'_2 = 1, \lambda'_3 = 0, \end{aligned}$$

$$\begin{aligned}
f(X^2) &= (1+X)\cdot 2X = 2X + 2X^2 \\
&= 0\cdot 1 + 2\cdot X + 2\cdot X^2 \\
&= \lambda_1''\cdot v_1 + \lambda_2''\cdot v_2 + \lambda_3''\cdot v_3 \\
&\Longrightarrow \lambda_2'' = \lambda_3'' = 2, \lambda_1'' = 0.
\end{aligned}$$

Thus, by Definition (2.13), the matrix of f with respect to $\mathcal{B}$ is,

$$[f]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \lambda_1 & \lambda_1' & \lambda_1'' \\ \lambda_2 & \lambda_2' & \lambda_2'' \\ \lambda_3 & \lambda_3' & \lambda_3'' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.29}$$

3. Matrix of f with respect to the basis $\mathcal{B}' = \{1, 1+X, (1+X)^2\}$

Method 1,

First, note that $(1+X)^2 = 1 + 2X + X^2$. So the basis vectors are,

$$b_1 = 1, \quad b_2 = 1 + X, \quad b_3 = 1 + 2X + X^2.$$

We need to compute $f(b_1)$, $f(b_2)$, $f(b_3)$ and express them in terms of $\mathcal{B}'$.

Compute,

$$\begin{aligned}
f(b_1) &= f(1) = 0 = 0\cdot b_1 + 0\cdot b_2 + 0\cdot b_3 \\
&= \lambda_1\cdot b_1 + \lambda_2\cdot b_2 + \lambda_3\cdot b_3 \\
&\Longrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0, \\
f(b_2) &= f(1+X) = (1+X)\cdot 1 = 1 + X \\
&= 0\cdot b_1 + 1\cdot b_2 + 0\cdot b_3 \\
&= \lambda_1'\cdot b_1 + \lambda_2'\cdot b_2 + \lambda_3'\cdot b_3 \\
&\Longrightarrow \lambda_1' = \lambda_3' = 0, \lambda_2' = 1, \\
f(b_3) &= f(1 + 2X + X^2) = (1+X)\cdot(2+2X) \\
&= 2(1+X)(1+X) \\
&= 2(1+X)^2 = 2b_3 \\
&= \lambda_1''\cdot b_1 + \lambda_2''\cdot b_2 + \lambda_3''\cdot b_3 \\
&\Longrightarrow \lambda_1'' = \lambda_2'' = 0, \lambda_3'' = 2.
\end{aligned}$$

Thus, the coordinates are,

- $f(b_1) = 0 = 0\cdot b_1 + 0\cdot b_2 + 0\cdot b_3$
- $f(b_2) = b_2 = 0\cdot b_1 + 1\cdot b_2 + 0\cdot b_3$
- $f(b_3) = 2b_3 = 0\cdot b_1 + 0\cdot b_2 + 2\cdot b_3$

Therefore, by Definition (2.13), the matrix of f with respect to $\mathcal{B}'$ is,

$$[f]_{\mathcal{B}'}^{\mathcal{B}'} = \begin{pmatrix} \lambda_1 & \lambda_1' & \lambda_1'' \\ \lambda_2 & \lambda_2' & \lambda_2'' \\ \lambda_3 & \lambda_3' & \lambda_3'' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2.30}$$

Method 2: Change-of-basis formula.

Let $\mathcal{B} = (1, X, X^2)$ be the canonical basis. We have from (2.29)

$$[f]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

First, find the change-of-basis matrix from $\mathcal{B}' = \{b_1, b_2, b_3\}$ to $\mathcal{B} = \{v_1, v_2, v_3\}$, denoted $P = [\mathrm{id}]_{\mathcal{B}'}^{\mathcal{B}}$ as in (2.14). The columns of P are the coordinates of the vectors of $\mathcal{B}'$ relative to $\mathcal{B}$.

We have,

$$\begin{aligned} b_1 = 1 &= 1 \cdot 1 + 0 \cdot X + 0 \cdot X^2 = 1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3, \\ 1 + X &= 1 \cdot 1 + 1 \cdot X + 0 \cdot X^2 = 1 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3, \\ (1+X)^2 = 1 + 2X + X^2 &= 1 \cdot 1 + 2 \cdot X + 1 \cdot X^2 = 1 \cdot v_1 + 2 \cdot v_2 + 1 \cdot v_3. \end{aligned}$$

Thus,

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Next, compute the inverse P^{-1} using the adjugate formula (2.11).

Since P is upper triangular, the determinant is the product of diagonal elements,

$$\det(P) = 1 \times 1 \times 1 = 1.$$

We compute the cofactors as in (2.6),

$$\begin{aligned} \mathrm{Cof}(P) &= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \\ &= \begin{pmatrix} +\det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} & -\det\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} & +\det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ -\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & +\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & -\det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ +\det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} & -\det\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} & +\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1\cdot(1\cdot 1-2\cdot 0) & -1\cdot(0\cdot 1-2\cdot 0) & 1\cdot(0\cdot 0-1\cdot 0)=0 \\ -1\cdot(1\cdot 1-1\cdot 0) & 1\cdot(1\cdot 1-1\cdot 0) & -1\cdot(1\cdot 0-1\cdot 0) \\ 1\cdot(1\cdot 2-1\cdot 1) & -1\cdot(1\cdot 2-1\cdot 0) & 1\cdot(1\cdot 1-1\cdot 0) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

Thus, the cofactor matrix is,

$$\mathrm{Cof}(P) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

The adjugate matrix is the transpose of the cofactor matrix,

$$\mathrm{adj}(P) = \mathrm{Cof}(P)^T = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\det(P) = 1$, we have,

$$P^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, by the change-of-basis formula for linear maps (2.15),

$$[f]_{\mathcal{B}'}^{\mathcal{B}'} = P^{-1}[f]_{\mathcal{B}}^{\mathcal{B}}P.$$

Compute $[f]_{\mathcal{B}}^{\mathcal{B}}P$,

$$[f]_{\mathcal{B}}^{\mathcal{B}}P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then multiply by P^{-1},

$$[f]_{\mathcal{B}'}^{\mathcal{B}'} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Thus, the matrix of f with respect to the basis $\mathcal{B}'$ is,

$$[f]_{\mathcal{B}'}^{\mathcal{B}'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

□

3. Solutions of Tutorial Exercises (3):

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3.1. Preliminary Notation

A system of m linear equations in n unknowns over a field $\mathbb{F}$ (typically $\mathbb{R}$ or $\mathbb{C}$) is a family of equations

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

where $a_{ij}, b_i \in \mathbb{F}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$,are given coefficients and constants, and $x_1, \ldots, x_n$ are the unknowns.

Definition 3.1 (Matrix Form):
The system can be written in matrix form as $A\mathbf{x} = \mathbf{b}$, where

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in M_{m\times n}(\mathbb{F}), \quad \mathbf{x} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n, \quad \mathbf{b} := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{F}^m.$$

Here A is the *coefficient matrix*, $\mathbf{x}$ the *unknown vector*, and $\mathbf{b}$ the *constant vector*.

***Remark* 3.1:**
When $m = n$, the system is called *square*. For a square system, if $\det A \neq 0$, then A is invertible and the system has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

3.2. Cramer's Rule

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in \mathcal{M}_n(\mathbb{F})$ be an invertible matrix ($\det A \neq 0$), and let $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{F}^n$.

For each $j = 1, \ldots, n$, denote by A_j the matrix obtained from A by replacing its jth column with $\mathbf{b}$,

$$A_j = \begin{pmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{pmatrix},$$

Then the unique solution $\mathbf{x} = (x_1, \ldots, x_n)^T$ of $A\mathbf{x} = \mathbf{b}$ is given by

$$x_j = \frac{\det A_j}{\det A}, \qquad j = 1, \ldots, n.$$

.

Example 3.1:
Consider the 3×3 linear system,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

The coefficient matrix and vectors are,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

with $\det(A) \neq 0$.

We construct,

$$A_1 = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}, \quad A_3 = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}$$

Using the formula for 3×3 determinants,

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33},$$

$$\begin{aligned}
\det(A_1) &= b_1a_{22}a_{33} + a_{12}a_{23}b_3 + a_{13}b_2a_{32} - a_{13}a_{22}b_3 - b_1a_{23}a_{32} - a_{12}b_2a_{33},\\
\det(A_2) &= a_{11}b_2a_{33} + b_1a_{23}a_{31} + a_{13}a_{21}b_3 - a_{13}b_2a_{31} - a_{11}a_{23}b_3 - b_1a_{21}a_{33},\\
\det(A_3) &= a_{11}a_{22}b_3 + a_{12}b_2a_{31} + b_1a_{21}a_{32} - b_1a_{22}a_{31} - a_{11}b_2a_{32} - a_{12}a_{21}b_3.
\end{aligned}$$

Then by Cramer's rule,

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad x_3 = \frac{\det(A_3)}{\det(A)}.$$

Exercise 3.1:
Solve the system of 2×2 linear equations using Cramer's Rule, if applicable

$$\begin{cases} 2x_1 + 5x_2 = 11, \\ -3x_1 + x_2 = -4, \end{cases} \qquad \begin{cases} 2x_1 + 6x_2 = 22 \\ -x_1 + 5x_2 = 53, \end{cases}$$

Solution:

- Let
$$A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ -4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$
First compute
$$\det(A) = (2)(1) - (5)(-3) = 2 + 15 = 17 \neq 0.$$
so Cramer's rule applies.

- Construct the modified matrices
$$A_1 = \begin{pmatrix} 11 & 5 \\ -4 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 11 \\ -3 & -4 \end{pmatrix}.$$
Their determinants are
$$\det(A_1) = (11)(1) - (5)(-4) = 11 + 20 = 31,$$
$$\det(A_2) = (2)(-4) - (11)(-3) = -8 + 33 = 25.$$

- Apply Cramer's Rule,
$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{31}{17}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{25}{17}.$$
Verification
$$2\left(\frac{31}{17}\right) + 5\left(\frac{25}{17}\right) = \frac{62 + 125}{17} = \frac{187}{17} = 11 \quad \checkmark,$$
$$-3\left(\frac{31}{17}\right) + \frac{25}{17} = \frac{-93 + 25}{17} = \frac{-68}{17} = -4 \quad \checkmark.$$

□

Solution:

- Define
$$A = \begin{pmatrix} 2 & 6 \\ -1 & 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 22 \\ 53 \end{pmatrix}.$$

Then

$$\det(A) = (2)(5) - (6)(-1) = 10 + 6 = 16 \neq 0.$$

Since $\det(A) \neq 0$, Cramer's Rule is applicable.

- Now form

$$A_1 = \begin{pmatrix} 22 & 6 \\ 53 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 22 \\ -1 & 53 \end{pmatrix}.$$

Compute

$$\det(A_1) = (22)(5) - (6)(53) = 110 - 318 = -208,$$
$$\det(A_2) = (2)(53) - (22)(-1) = 106 + 22 = 128.$$

- Apply Cramer's Rule,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-208}{16} = -13, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{128}{16} = 8.$$

Verification

$$2(-13) + 6(8) = -26 + 48 = 22 \quad \checkmark,$$
$$-(-13) + 5(8) = 13 + 40 = 53 \quad \checkmark.$$

□

Exercise 3.2:
Solve the system of 3×3 linear equations using Cramer's Rule, if applicable

$$\begin{cases} 2x_1 + 3x_2 - x_3 = -12, \\ x_1 - x_2 - x_3 = -4, \\ -4x_1 + 3x_2 + x_3 = 14, \end{cases} \qquad \begin{cases} 5x_1 - 2x_2 + 3x_3 = 16, \\ 2x_1 + 3x_2 - 5x_3 = 2, \\ 4x_1 - 5x_2 + 6x_3 = 7. \end{cases}$$

Solution:

- The coefficient matrix and constant vector are,

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & -1 \\ -4 & 3 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -12 \\ -4 \\ 14 \end{pmatrix}.$$

Compute $\det A$ by expansion along the first ro,

$$\begin{aligned} \det(A) &= 2\det\begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix} - 3\det\begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix} + (-1)\det\begin{pmatrix} 1 & -1 \\ -4 & 3 \end{pmatrix} \\ &= 2[(-1)(1) - (-1)(3)] - 3[(1)(1) - (-1)(-4)] - [(1)(3) - (-1)(-4)] \\ &= 2(-1 + 3) - 3(1 - 4) - (3 - 4) \\ &= 2(2) - 3(-3) - (-1) = 4 + 9 + 1 = 14 \neq 0. \end{aligned}$$

Since $\det(A) \neq 0$, Cramer's Rule is applicable.

- Define the modified matrices,

$$A_1 = \begin{pmatrix} -12 & 3 & -1 \\ -4 & -1 & -1 \\ 14 & 3 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & -12 & -1 \\ 1 & -4 & -1 \\ -4 & 14 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 3 & -12 \\ 1 & -1 & -4 \\ -4 & 3 & 14 \end{pmatrix}.$$

Compute each determinant via cofactor expansion,

$$\begin{aligned} \det(A_1) &= (-12)\det\begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix} - 3\det\begin{pmatrix} -4 & -1 \\ 14 & 1 \end{pmatrix} + (-1)\det\begin{pmatrix} -4 & -1 \\ 14 & 3 \end{pmatrix} \\ &= (-12)[(-1)(1) - (-1)(3)] - 3[(-4)(1) - (-1)(14)] - [(-4)(3) - (-1)(14)] \\ &= (-12)(-1+3) - 3(-4+14) - (-12+14) \\ &= (-12)(2) - 3(10) - (2) = -24 - 30 - 2 = -56. \end{aligned}$$

$$\begin{aligned} \det(A_2) &= 2\det\begin{pmatrix} -4 & -1 \\ 14 & 1 \end{pmatrix} - (-12)\det\begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix} + (-1)\det\begin{pmatrix} 1 & -4 \\ -4 & 14 \end{pmatrix} \\ &= 2[(-4)(1) - (-1)(14)] + 12[(1)(1) - (-1)(-4)] - [(1)(14) - (-4)(-4)] \\ &= 2(-4+14) + 12(1-4) - (14-16) \\ &= 2(10) + 12(-3) - (-2) = 20 - 36 + 2 = -14. \end{aligned}$$

$$\begin{aligned} \det(A_3) &= 2\det\begin{pmatrix} -1 & -4 \\ 3 & 14 \end{pmatrix} - 3\det\begin{pmatrix} 1 & -4 \\ -4 & 14 \end{pmatrix} + (-12)\det\begin{pmatrix} 1 & -1 \\ -4 & 3 \end{pmatrix} \\ &= 2[(-1)(14) - (-4)(3)] - 3[(1)(14) - (-4)(-4)] - 12[(1)(3) - (-1)(-4)] \\ &= 2(-14+12) - 3(14-16) - 12(3-4) \\ &= 2(-2) - 3(-2) - 12(-1) = -4 + 6 + 12 = 14. \end{aligned}$$

- Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-56}{14} = -4, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-14}{14} = -1, \quad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{14}{14} = 1.$$

Verification

$$\begin{aligned} 2(-4) + 3(-1) - 1 = -8 - 3 - 1 = -12 \quad \checkmark, \\ -4 - (-1) - 1 = -4 + 1 - 1 = -4 \quad \checkmark, \\ -4(-4) + 3(-1) + 1 = 16 - 3 + 1 = 14 \quad \checkmark. \end{aligned}$$

$\square$

Solution:

- Set
$$A = \begin{pmatrix} 5 & -2 & 3 \\ 2 & 3 & -5 \\ 4 & -5 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 16 \\ 2 \\ 7 \end{pmatrix}.$$
Compute $\det(A)$ using cofactor expansion along first row,
$$\begin{aligned} \det(A) &= 5 \det \begin{pmatrix} 3 & -5 \\ -5 & 6 \end{pmatrix} - (-2) \det \begin{pmatrix} 2 & -5 \\ 4 & 6 \end{pmatrix} + 3 \det \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix} \\ &= 5[(3)(6) - (-5)(-5)] + 2[(2)(6) - (-5)(4)] + 3[(2)(-5) - (3)(4)] \\ &= 5(18 - 25) + 2(12 + 20) + 3(-10 - 12) \\ &= 5(-7) + 2(32) + 3(-22) = -35 + 64 - 66 = -37 \neq 0. \end{aligned}$$
Since $\det(A) \neq 0$, Cramer's Rule is applicable.

- Define
$$A_1 = \begin{pmatrix} 16 & -2 & 3 \\ 2 & 3 & -5 \\ 7 & -5 & 6 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 5 & 16 & 3 \\ 2 & 2 & -5 \\ 4 & 7 & 6 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 5 & -2 & 16 \\ 2 & 3 & 2 \\ 4 & -5 & 7 \end{pmatrix}.$$
Compute determinants,
$$\begin{aligned} \det(A_1) &= 16 \det \begin{pmatrix} 3 & -5 \\ -5 & 6 \end{pmatrix} - (-2) \det \begin{pmatrix} 2 & -5 \\ 7 & 6 \end{pmatrix} + 3 \det \begin{pmatrix} 2 & 3 \\ 7 & -5 \end{pmatrix} \\ &= 16[(3)(6) - (-5)(-5)] + 2[(2)(6) - (-5)(7)] + 3[(2)(-5) - (3)(7)] \\ &= 16(18 - 25) + 2(12 + 35) + 3(-10 - 21) \\ &= 16(-7) + 2(47) + 3(-31) = -112 + 94 - 93 = -111. \end{aligned}$$
$$\begin{aligned} \det(A_2) &= 5 \det \begin{pmatrix} 2 & -5 \\ 7 & 6 \end{pmatrix} - 16 \det \begin{pmatrix} 2 & -5 \\ 4 & 6 \end{pmatrix} + 3 \det \begin{pmatrix} 2 & 2 \\ 4 & 7 \end{pmatrix} \\ &= 5[(2)(6) - (-5)(7)] - 16[(2)(6) - (-5)(4)] + 3[(2)(7) - (2)(4)] \\ &= 5(12 + 35) - 16(12 + 20) + 3(14 - 8) \\ &= 5(47) - 16(32) + 3(6) = 235 - 512 + 18 = -259. \end{aligned}$$
$$\begin{aligned} \det(A_3) &= 5 \det \begin{pmatrix} 3 & 2 \\ -5 & 7 \end{pmatrix} - (-2) \det \begin{pmatrix} 2 & 2 \\ 4 & 7 \end{pmatrix} + 16 \det \begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix} \\ &= 5[(3)(7) - (2)(-5)] + 2[(2)(7) - (2)(4)] + 16[(2)(-5) - (3)(4)] \\ &= 5(21 + 10) + 2(14 - 8) + 16(-10 - 12) \\ &= 5(31) + 2(6) + 16(-22) = 155 + 12 - 352 = -185. \end{aligned}$$

- Thus
$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-111}{-37} = 3, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-259}{-37} = 7, \quad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-185}{-37} = 5.$$
Verification
$$5(3) - 2(7) + 3(5) = 15 - 14 + 15 = 16 \quad \checkmark,$$

$$2(3) + 3(7) - 5(5) = 6 + 21 - 25 = 2 \quad \checkmark,$$
$$4(3) - 5(7) + 6(5) = 12 - 35 + 30 = 7 \quad \checkmark.$$

$\square$

4. Solutions of Tutorial Exercises (4):

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4.1. Definitions and Notation

Let $\mathbb{F}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and let A be an $n \times n$ matrix with entries in $\mathbb{F}$.

Definition 4.1 (Eigenvalues and Eigenvectors):
A scalar $\lambda \in \mathbb{F}$ is called an *eigenvalue* of A if there exists a non-zero vector $\mathbf{v} \in \mathbb{F}^n$ such that,

$$A\mathbf{v} = \lambda\mathbf{v}.$$

The vector $\mathbf{v}$ is called an *eigenvector* corresponding to λ.

Definition 4.2 (Spectrum):
The set of all eigenvalues of a matrix A is called its *spectrum*, denoted by,

$$\mathrm{Spec}(A) = \{\lambda \in \mathbb{F} : \exists \mathbf{v} \neq \mathbf{0} \text{ such that } A\mathbf{v} = \lambda\mathbf{v}\}.$$

Definition 4.3 (Eigenspace):
For an eigenvalue $\lambda \in \mathrm{Spec}(A)$, the *eigenspace* associated with λ is,

$$E_\lambda(A) = \mathrm{Ker}(A - \lambda I_n) = \{\mathbf{v} \in \mathbb{F}^n : (A - \lambda I_n)\mathbf{v} = \mathbf{0}\}.$$

This is a subspace of $\mathbb{F}^n$ containing all eigenvectors for λ together with the zero vector.

Definition 4.4 (Characteristic Polynomial):
The *characteristic polynomial* of an $n \times n$ matrix A is,

$$p_A(\lambda) = \det(A - \lambda I_n).$$

This is a monic polynomial of degree n in λ.

By the Fundamental Theorem of Algebra, when $\mathbb{F} = \mathbb{C}$, $p_A(\lambda)$ factors completely as

$$p_A(\lambda) = (-1)^n \prod_{i=1}^{k} (\lambda - \lambda_i)^{m_i},$$

where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues, and m_i are positive integers summing to n.

4.2. Multiplicities, Diagonalizability and Trigonalizability

Definition 4.5 (Algebraic Multiplicity):
The *algebraic multiplicity* of an eigenvalue $\lambda \in \operatorname{Spec}(A)$, denoted $a_A(\lambda)$, is its multiplicity as a root of the characteristic polynomial $p_A(\lambda)$.

Definition 4.6 (Geometric Multiplicity):
The *geometric multiplicity* of $\lambda \in \operatorname{Spec}(A)$, denoted $g_A(\lambda)$, is,

$$g_A(\lambda) = \dim E_\lambda(A) = \underbrace{\operatorname{Nullity}(A - \lambda I_n)}_{\dim \operatorname{Ker}(A-\lambda I_n)} = n - \underbrace{\operatorname{Rank}(A - \lambda I_n)}_{\dim(\operatorname{Im}(A-\lambda I_n))}.$$

Theorem 4.1 (Relation Between Multiplicities):
For any eigenvalue $\lambda \in \operatorname{Spec}(A)$, we have,

$$1 \leq g_A(\lambda) \leq a_A(\lambda) \leq n.$$

Definition 4.7 (Diagonalizability):
An $n \times n$ matrix A is *diagonalizable* over $\mathbb{F}$ if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{F})$ such that,

$$P^{-1}AP = \operatorname{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A (not necessarily distinct).

Equivalently, A is diagonalizable if and only if $\mathbb{F}^n$ has a basis consisting entirely of eigenvectors of A.

Theorem 4.2:
Let $A \in \mathcal{M}_n(\mathbb{C})$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct eigenvalues of A, then A is diagonalizable.

Theorem 4.3 (Diagonalizability Criterion):
A matrix $A \in \mathcal{M}_n(\mathbb{C})$ is diagonalizable over $\mathbb{C}$ if and only if for every eigenvalue $\lambda \in \operatorname{Spec}(A)$, we have,

$$g_A(\lambda) = a_A(\lambda).$$

Moreover, if A is diagonalizable, then,

$$\mathbb{C}^n = \bigoplus_{\lambda \in \operatorname{Spec}(A)} E_\lambda(A),$$

where the direct sum is taken over all distinct eigenvalues.

Theorem 4.4 (Trace and Determinant Relations):
Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (counted with algebraic multiplicity). Then,

$$\operatorname{Trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Definition 4.8 (Trigonalizable Matrix):
A matrix $A \in \mathcal{M}_n(\mathbb{C})$ is *trigonalizable* (or *triangularizable*) if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{C})$ such that $P^{-1}AP$ is upper triangular.

Theorem 4.5 (Trigonalizability Theorem):
Every complex square matrix is trigonalizable. That is, for any $A \in \mathcal{M}_n(\mathbb{C})$, there exists an invertible matrix P such that,

$$P^{-1}AP = T = \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A (not necessarily distinct).

***Remark* 4.1 (Trigonalizability vs Diagonalizability):**

- *Diagonalizable* $\Rightarrow$ Trigonalizable (since diagonal matrices are a special case of triangular matrices)
- *Trigonalizable* $\not\Rightarrow$ *Diagonalizable* (e.g., matrix B_4 in the exercises)
- Over $\mathbb{C}$, all matrices are trigonalizable, but not all are diagonalizable
- Over $\mathbb{R}$, a matrix is trigonalizable only if all its eigenvalues are real (otherwise, complex eigenvalues are needed)

Theorem 4.6 (Trigonalization via Basis Completion):
Any matrix $A \in \mathcal{M}_n(\mathbb{C})$ can be trigonalized by completing a basis of $\mathbb{C}^n$ to an A-invariant flag. That is, there exists a basis $\{v_1, v_2, \ldots, v_n\}$ such that for each $k = 1, \ldots, n$,

$$Av_k \in \operatorname{Span}\{v_1, v_2, \ldots, v_k\}$$

Computational Methods for Trigonalization

To trigonalize a matrix $A \in \mathcal{M}_n(\mathbb{C})$ (i.e., put it in upper triangular form), follow these steps,

1. Compute eigenvalues: Find all eigenvalues $\lambda_1, \ldots, \lambda_n$ of A with their algebraic multiplicities.

2. Find eigenvectors and generalized eigenvectors: For each eigenvalue λ with algebraic multiplicity m,

   a. Find ordinary eigenvectors by solving $(A - \lambda I)v = 0$.

   b. If the number of linearly independent eigenvectors is less than m, find generalized eigenvectors by solving,

- $(A - \lambda I)^2 v = 0$ for vectors not in $\text{Ker}(A - \lambda I)$
- Continue until $(A - \lambda I)^k v = 0$ for the smallest k giving new vectors
- Build Jordan chains: if $(A - \lambda I)v_{i+1} = v_i$, then v_{i+1} is a generalized eigenvector

3. Construct Jordan chains: For each eigenvalue, arrange vectors in chains,

$$v_1, v_2, \ldots, v_p \quad \text{with} \quad (A - \lambda I)v_1 = 0,\ (A - \lambda I)v_2 = v_1,\ \ldots,\ (A - \lambda I)v_p = v_{p-1}$$

4. Order the basis: Arrange all vectors (from all eigenvalues) in a single ordered basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ such that,
   - For each eigenvalue, complete its chain before moving to the next eigenvalue
   - Typically order eigenvalues by increasing real part or magnitude
   - Within each chain, order vectors from "most eigen" to "most generalized"

5. Form the change-of-basis matrix: Construct P as the matrix whose columns are the vectors of $\mathcal{B}$,

$$P = [v_1\ v_2\ \cdots\ v_n]$$

6. Compute the triangular form: Calculate $T = P^{-1}AP$, which will be upper triangular.

Exercise 4.1 (Spectral Analysis of Four Matrices):
Consider the following matrices over $\mathbb{C}$,

$$B_1 = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}.$$

For each matrix B_i $(i = 1, 2, 3, 4)$,

(1) Compute the characteristic polynomial $p_{B_i}(\lambda) = \det(B_i - \lambda I)$ and express it in fully factorized form.

(2) Determine all eigenvalues $\lambda \in \operatorname{Spec}(B_i)$. For each eigenvalue, compute its algebraic multiplicity $a_{B_i}(\lambda)$.

(3) For each eigenvalue λ, find a basis of the eigenspace

$$E_\lambda(B_i) = \operatorname{Ker}(B_i - \lambda I) = \{\mathbf{v} \in \mathbb{C}^3 : (B_i - \lambda I_3)\mathbf{v} = \mathbf{0}\}$$

and compute its geometric multiplicity

$$g_{B_i}(\lambda) = \dim E_\lambda(B_i).$$

(4) If B_i is diagonalizable, determine a diagonal matrix $D_i = P^{-1} B_i P$ that is similar to B_i.

Solution:

Matrix $B_1 = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

(1) Characteristic Polynomial

$$\begin{aligned} p_{B_1}(\lambda) &= \det(B_1 - \lambda I_2) \\ &= \det\begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} \\ &= (1-\lambda)(4-\lambda) - (-1)(2) \\ &= (1-\lambda)(4-\lambda) + 2 \\ &= 4 - 4\lambda - \lambda + \lambda^2 + 2 \\ &= \lambda^2 - 5\lambda + 6. \end{aligned}$$

To factor this quadratic polynomial, we solve $\lambda^2 - 5\lambda + 6 = 0$,

$$\lambda = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2} = \begin{cases} \lambda_1 = 3, \\ \lambda_2 = 2. \end{cases}$$

Thus, the characteristic polynomial in fully factored form is,

$$p_{B_1}(\lambda) = (\lambda - 2)(\lambda - 3).$$

(2) Eigenvalues and Their Multiplicities

- $\lambda_1 = 2$: Algebraic multiplicity $a_{B_1}(2) = 1$
- $\lambda_2 = 3$: Algebraic multiplicity $a_{B_1}(3) = 1$

(3) Eigenvectors and Geometric Multiplicities

- **For $\lambda = 2$,**

$$B_1 - 2I = \begin{pmatrix} 1-2 & -1 \\ 2 & 4-2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}.$$

We solve $(B_1 - 2I)\mathbf{v} = \mathbf{0}$,

$$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the equation $-v_1 - v_2 = 0$, or $v_1 = -v_2$. So,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_1 \in \mathbb{C}.$$

Thus, $E_2(B_1) = \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ and $g_{B_1}(2) = \dim E_2(B_1) = 1$.

- **For $\lambda = 3$,**

$$B_1 - 3I = \begin{pmatrix} 1-3 & -1 \\ 2 & 4-3 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}.$$

We solve $(B_1 - 3I)\mathbf{v} = \mathbf{0}$,

$$\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the equation $-2v_1 - v_2 = 0$, or $v_2 = -2v_1$. So,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad v_1 \in \mathbb{C}.$$

Thus, $E_3(B_1) = \text{Span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$ and $g_{B_1}(3) = \dim E_3(B_1) = 1$.

(4) Diagonalizability Check

- For $\lambda = 2$: $a_{B_1}(2) = 1 = g_{B_1}(2)$
- For $\lambda = 3$: $a_{B_1}(3) = 1 = g_{B_1}(3)$

Since $g_{B_1}(\lambda) = a_{B_1}(\lambda)$ for all eigenvalues, B_1 is diagonalizable.

a) Diagonalization Matrix

We can construct P_1 using the eigenvectors as columns,

$$P_1 = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}.$$

Then,

$$D_1 = P_1^{-1}B_1P_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Matrix $B_2 = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

1. Characteristic Polynomial

$$\begin{aligned} p_{B_2}(\lambda) &= \det(B_2 - \lambda I_3) \\ &= \det \begin{pmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix}. \end{aligned}$$

We compute this determinant using cofactor expansion along the first row,

$$\begin{aligned} p_{B_2}(\lambda) &= (-\lambda) \cdot \det \begin{pmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} - 0 \cdot \det \begin{pmatrix} 1 & 1 \\ 1 & 3-\lambda \end{pmatrix} + (-2) \cdot \det \begin{pmatrix} 1 & 2-\lambda \\ 1 & 0 \end{pmatrix} \\ &= (-\lambda) \cdot [(2-\lambda)(3-\lambda) - 1 \cdot 0] - 0 - 2 \cdot [1 \cdot 0 - (2-\lambda) \cdot 1] \\ &= (-\lambda)(2-\lambda)(3-\lambda) - 2[-(2-\lambda)] \\ &= -\lambda(2-\lambda)(3-\lambda) + 2(2-\lambda) \\ &= (2-\lambda)[-\lambda(3-\lambda) + 2] \\ &= (2-\lambda)[-\lambda(3-\lambda) + 2] \\ &= (2-\lambda)[-3\lambda + \lambda^2 + 2] \\ &= (2-\lambda)(\lambda^2 - 3\lambda + 2) \\ &= (2-\lambda)(\lambda-1)(\lambda-2) \\ &= -(\lambda-2)^2(\lambda-1). \end{aligned}$$

Thus, the characteristic polynomial in fully factored form is,

$$p_{B_2}(\lambda) = -(\lambda-2)^2(\lambda-1).$$

2. Eigenvalues and Their Multiplicities

- $\lambda_1 = 1$: Algebraic multiplicity $a_{B_2}(1) = 1$
- $\lambda_2 = 2$: Algebraic multiplicity $a_{B_2}(2) = 2$

3. Eigenvectors and Geometric Multiplicities

- **For** $\lambda = 1$**,**

$$B_2 - I = \begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

We solve $(B_2 - I)\mathbf{v} = \mathbf{0}$ by row reduction,

$$\begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{R_1' \leftarrow -R_1} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow[R_3' \leftarrow R_3 - R_1']{R_2' \leftarrow R_2 - R_1'} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives the system,

$$\begin{aligned} v_1 + 2v_3 = 0 &\Rightarrow v_1 = -2v_3, \\ v_2 - v_3 = 0 &\Rightarrow v_2 = v_3. \end{aligned}$$

So,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -2v_3 \\ v_3 \\ v_3 \end{pmatrix}, = v_3 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 \in \mathbb{C}.$$

Thus, $E_1(B_2) = \operatorname{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right\}$ and $g_{B_2}(1) = \dim E_1(B_2) = 1$.

- **For $\lambda = 2$,**

$$B_2 - 2I = \begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

We solve $(B_2 - 2I)\mathbf{v} = \mathbf{0}$ by row reduction,

$$\begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_1' \leftarrow -\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \xrightarrow[R_3' \leftarrow R_3 - R_1']{R_2' \leftarrow R_2 - R_1'} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives the equation $v_1 + v_3 = 0$, or $v_1 = -v_3$. The variables v_2 and v_3 are free (with v_3 determining v_1). So,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_3 \\ v_2 \\ v_3 \end{pmatrix}, = v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2, v_3 \in \mathbb{C}.$$

Linear independence check

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that,

$$\lambda_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives,

$$\begin{cases} \lambda_1 = 0 \\ -\lambda_2 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = 0.$$

- Thus, $E_2(B_2) = \text{Span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ and $g_{B_2}(2) = \dim E_2(B_2) = 2$.

4. Diagonalizability Check

- For $\lambda = 1$: $a_{B_2}(1) = 1 = g_{B_2}(1)$
- For $\lambda = 2$: $a_{B_2}(2) = 2 = g_{B_2}(2)$

Since $g_{B_2}(\lambda) = a_{B_2}(\lambda)$ for all eigenvalues, B_2 is diagonalizable.

a) Diagonalization Matrix

We can construct P_2 using the eigenvectors as columns,

$$P_2 = \begin{pmatrix}-2 & 0 & -1\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}.$$

Then,

$$D_2 = P_2^{-1} B_2 P_2 = \begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_2\end{pmatrix}.$$

Matrix $B_3 = \begin{pmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{pmatrix}$

1. Characteristic Polynomial

$$\begin{aligned} p_{B_3}(\lambda) &= \det(B_3 - \lambda I_3) \\ &= \det \begin{pmatrix} 4-\lambda & 0 & 1 \\ -1 & -6-\lambda & -2 \\ 5 & 0 & -\lambda \end{pmatrix}. \end{aligned}$$

We compute this determinant using cofactor expansion along the second column (since it contains two zeros, which simplifies computation),

$$\begin{aligned} p_{B_3}(\lambda) &= 0 \cdot C_{12} + (-6-\lambda) \cdot C_{22} + 0 \cdot C_{32} \\ &= (-6-\lambda) \cdot (-1)^{2+2} \cdot \det \begin{pmatrix} 4-\lambda & 1 \\ 5 & -\lambda \end{pmatrix} \\ &= (-6-\lambda) \cdot \det \begin{pmatrix} 4-\lambda & 1 \\ 5 & -\lambda \end{pmatrix} \\ &= (-6-\lambda) \cdot [(4-\lambda)(-\lambda) - 1 \cdot 5] \\ &= (-6-\lambda) \cdot [-\lambda(4-\lambda) - 5] \\ &= (-6-\lambda) \cdot [-4\lambda + \lambda^2 - 5] \\ &= (-6-\lambda) \cdot (\lambda^2 - 4\lambda - 5) \\ &= (-6-\lambda) \cdot (\lambda-5)(\lambda+1) \quad (\text{since } \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1)) \\ &= -(\lambda+6)(\lambda-5)(\lambda+1). \end{aligned}$$

Thus, the characteristic polynomial in fully factored form is,

$$p_{B_3}(\lambda) = -(\lambda+6)(\lambda-5)(\lambda+1).$$

2. Eigenvalues and Their Multiplicities

- $\lambda_1 = -6$: Algebraic multiplicity $a_{B_3}(-6) = 1$
- $\lambda_2 = 5$: Algebraic multiplicity $a_{B_3}(5) = 1$
- $\lambda_3 = -1$: Algebraic multiplicity $a_{B_3}(-1) = 1$

3. Eigenvectors and Geometric Multiplicities

- **For** $\lambda = -6$**,**

$$B_3 + 6I = \begin{pmatrix} 10 & 0 & 1 \\ -1 & 0 & -2 \\ 5 & 0 & 6 \end{pmatrix}.$$

We solve $(B_3+6I)\mathbf{v}=\mathbf{0}$ by row reduction,

$$\begin{pmatrix}10 & 0 & 1\\ -1 & 0 & -2\\ 5 & 0 & 6\end{pmatrix} \xrightarrow{R_1'\leftarrow\frac{1}{10}R_1} \begin{pmatrix}1 & 0 & \frac{1}{10}\\ -1 & 0 & -2\\ 5 & 0 & 6\end{pmatrix} \xrightarrow[R_3'\leftarrow R_3-5R_1']{R_2'\leftarrow R_2+R_1'} \begin{pmatrix}1 & 0 & \frac{1}{10}\\ 0 & 0 & -\frac{19}{10}\\ 0 & 0 & \frac{11}{2}\end{pmatrix}$$

$$\xrightarrow{R_2''\leftarrow-\frac{10}{19}R_2'} \begin{pmatrix}1 & 0 & \frac{1}{10}\\ 0 & 0 & 1\\ 0 & 0 & \frac{11}{2}\end{pmatrix} \xrightarrow[R_3''\leftarrow R_3'-\frac{11}{2}R_2'']{R_1''\leftarrow R_1'-\frac{1}{10}R_2''} \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}.$$

This gives the system,

$$v_1=0,$$
$$v_3=0,$$

with v_2 free. So,

$$\mathbf{v}=\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}0\\ v_2\\ 0\end{pmatrix},=v_2\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}, \quad v_2\in\mathbb{C}.$$

Thus, $E_{-6}(B_3)=\text{Span}\left\{\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}\right\}$ and $g_{B_3}(-6)=\dim E_{-6}(B_3)=1$.

- **For $\lambda=5$,**

$$B_3-5I=\begin{pmatrix}-1 & 0 & 1\\ -1 & -11 & -2\\ 5 & 0 & -5\end{pmatrix}.$$

We solve $(B_3-5I)\mathbf{v}=\mathbf{0}$ by row reduction,

$$\begin{pmatrix}-1 & 0 & 1\\ -1 & -11 & -2\\ 5 & 0 & -5\end{pmatrix} \xrightarrow{R_1'\leftarrow-R_1} \begin{pmatrix}1 & 0 & -1\\ -1 & -11 & -2\\ 5 & 0 & -5\end{pmatrix} \xrightarrow[R_3'\leftarrow R_3-5R_1']{R_2'\leftarrow R_2+R_1'} \begin{pmatrix}1 & 0 & -1\\ 0 & -11 & -3\\ 0 & 0 & 0\end{pmatrix}$$

$$\xrightarrow{R_2''\leftarrow-\frac{1}{11}R_2'} \begin{pmatrix}1 & 0 & -1\\ 0 & 1 & \frac{3}{11}\\ 0 & 0 & 0\end{pmatrix}.$$

This gives the system,

$$v_1-v_3=0\Rightarrow v_1=v_3,$$
$$v_2+\frac{3}{11}v_3=0\Rightarrow v_2=-\frac{3}{11}v_3.$$

So,

$$\mathbf{v}=\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}=\begin{pmatrix}v_3\\ -\frac{3}{11}v_3\\ v_3\end{pmatrix},=v_3\begin{pmatrix}1\\ -\frac{3}{11}\\ 1\end{pmatrix}, \quad v_3\in\mathbb{C}.$$

To get integer coordinates, multiply by 11,

$$\mathbf{v} = v_3 \begin{pmatrix} 11 \\ -3 \\ 11 \end{pmatrix}, \quad v_3 \in \mathbb{C}.$$

Thus, $E_5(B_3) = \text{Span}\left\{\begin{pmatrix} 11 \\ -3 \\ 11 \end{pmatrix}\right\}$ and $g_{B_3}(5) = \dim E_5(B_3) = 1$.

- **For $\lambda = -1$,**

$$B_3 + I = \begin{pmatrix} 5 & 0 & 1 \\ -1 & -5 & -2 \\ 5 & 0 & 1 \end{pmatrix}.$$

We solve $(B_3 + I)\mathbf{v} = \mathbf{0}$ by row reduction,

$$\begin{pmatrix} 5 & 0 & 1 \\ -1 & -5 & -2 \\ 5 & 0 & 1 \end{pmatrix} \xrightarrow{R_1' \leftarrow \frac{1}{5}R_1} \begin{pmatrix} 1 & 0 & \frac{1}{5} \\ -1 & -5 & -2 \\ 5 & 0 & 1 \end{pmatrix} \xrightarrow[R_3' \leftarrow R_3 - 5R_1']{R_2' \leftarrow R_2 + R_1'} \begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & -5 & -\frac{9}{5} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2'' \leftarrow -\frac{1}{5}R_2'} \begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{9}{25} \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives the system,

$$v_1 + \frac{1}{5}v_3 = 0 \Rightarrow v_1 = -\frac{1}{5}v_3,$$
$$v_2 + \frac{19}{25}v_3 = 0 \Rightarrow v_2 = -\frac{9}{25}v_3.$$

So,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5}v_3 \\ -\frac{9}{25}v_3 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} -\frac{1}{5} \\ -\frac{9}{25} \\ 1 \end{pmatrix}, \quad v_3 \in \mathbb{C}.$$

Multiply by -25,

$$\mathbf{v} = v_3 \begin{pmatrix} 5 \\ 9 \\ -25 \end{pmatrix}, \quad v_3 \in \mathbb{C}.$$

Thus, $E_{-1}(B_3) = \text{Span}\left\{\begin{pmatrix} 5 \\ 9 \\ -25 \end{pmatrix}\right\}$ and $g_{B_3}(-1) = \dim E_{-1}(B_3) = 1$.

4. Diagonalizability Check

- For $\lambda = -6$: $a_{B_3}(-6) = 1 = g_{B_3}(-6)$

- For $\lambda = 5$: $a_{B_3}(5) = 1 = g_{B_3}(5)$

- For $\lambda = -1$: $a_{B_3}(-1) = 1 = g_{B_3}(-1)$

Since $g_{B_3}(\lambda) = a_{B_3}(\lambda)$ for all eigenvalues, B_3 is diagonalizable.

5. Diagonalization Matrix

We can construct P_3 using the eigenvectors as columns,

$$P_3 = \begin{pmatrix} 0 & 11 & 5 \\ 1 & -3 & 9 \\ 0 & 11 & -25 \end{pmatrix}.$$

Then,

$$P_3^{-1} B_3 P_3 = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Matrix $B_4 = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$

1. Characteristic Polynomial

   Since B_4 is upper triangular, its eigenvalues are exactly its diagonal entries. The characteristic polynomial is,

$$\begin{aligned} p_{B_4}(\lambda) &= \det(B_4 - \lambda I_3) \\ &= \det \begin{pmatrix} 2-\lambda & 4 & 6 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 4-\lambda \end{pmatrix} \\ &= (2-\lambda)(2-\lambda)(4-\lambda) \quad \text{(product of diagonal entries)} \\ &= (2-\lambda)^2(4-\lambda). \end{aligned}$$

   Thus, the characteristic polynomial in fully factored form is,

$$p_{B_4}(\lambda) = (2-\lambda)^2(4-\lambda).$$

2. Eigenvalues and Their Multiplicities

   - $\lambda_1 = 2$: Algebraic multiplicity $a_{B_4}(2) = 2$
   - $\lambda_2 = 4$: Algebraic multiplicity $a_{B_4}(4) = 1$

3. Eigenvectors and Geometric Multiplicities

   - **For $\lambda = 2$,**

$$B_4 - 2I = \begin{pmatrix} 0 & 4 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

     We solve $(B_4 - 2I)\mathbf{v} = \mathbf{0}$ by row reduction,

$$\begin{pmatrix} 0 & 4 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_3' \leftarrow R_3 - R_2} \begin{pmatrix} 0 & 4 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2' \leftarrow \frac{1}{2} R_2]{R_1' \leftarrow \frac{1}{4} R_1} \begin{pmatrix} 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1'' \leftarrow R_1' - \frac{3}{2} R_2'} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

     This gives the system,

$$\begin{aligned} v_2 &= 0, \\ v_3 &= 0, \end{aligned}$$

with v_1 free. So,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_1 \in \mathbb{C}.$$

Thus, $E_2(B_4) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ and $g_{B_4}(2) = \dim E_2(B_4) = 1$.

- **For $\lambda = 4$,**

$$B_4 - 4I = \begin{pmatrix} -2 & 4 & 6 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We solve $(B_4 - 4I)\mathbf{v} = \mathbf{0}$ by row reduction,

$$\begin{pmatrix} -2 & 4 & 6 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2' \leftarrow -\frac{1}{2}R_2]{R_1' \leftarrow -\frac{1}{2}R_1} \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1'' \leftarrow R_1' + 2R_2'} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives the system,

$$v_1 - 5v_3 = 0 \Rightarrow v_1 = 5v_3,$$
$$v_2 - v_3 = 0 \Rightarrow v_2 = v_3.$$

So,

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 5v_3 \\ v_3 \\ v_3 \end{pmatrix} = v_3 \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 \in \mathbb{C}.$$

Thus, $E_4(B_4) = \text{Span}\left\{ \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $g_{B_4}(4) = \dim E_4(B_4) = 1$.

4. Diagonalizability Check

- For $\lambda = 2$: $a_{B_4}(2) = 2$, but $g_{B_4}(2) = 1$
- For $\lambda = 4$: $a_{B_4}(4) = 1 = g_{B_4}(4)$

Since $g_{B_4}(2) < a_{B_4}(2)$, B_4 is **not** diagonalizable.

b) Why B_4 is Not Diagonalizable

For B_4 to be diagonalizable, we would need three linearly independent eigenvectors (since it's a 3×3 matrix). However, we only found,

- One eigenvector for $\lambda = 2$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

- One eigenvector for $\lambda = 4$, $\begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}$

This gives only two linearly independent eigenvectors, which is insufficient to form a basis for $\mathbb{C}^3$.

$\square$

Solutions of Tutorial Exercises

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